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Non-linear strategies in a linear quadratic differential game

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Abstract

We study non-linear Markov perfect equilibria in a two agent linear quadratic differential game. In contrast to the literature owing to Tsutsui and Mino [1990. Nonlinear strategies in dynamic duopolistic competition with sticky prices. Journal of Economic Theory 52, 136–161], we do not associate endogenous subsets of the state space with candidate solutions. Instead, we use the 'catching up optimality' criterion to address the possibility of infinitely valued value functions. Applying sufficiency conditions for existence based on those in Dockner et al. [2000. Differential Games in Economics and Management Science. Cambridge University Press, Cambridge] yields the familiar linear MPE and a condition under which a continuum of non-linear MPEs exists. These include, as their limit, a previously unreported piecewise linear MPE. The condition relaxes with increasing patience, allowing more efficient steady states, thus suggesting a Folk Theorem for differential games. As the lower state and control bounds go to $-\infty$, the non-linear strategies are eliminated.

JEL classification: C61; C73; H41; Q00

Keywords: Differential game; Non-linear strategies; Catching up optimal; Folk Theorem

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1. Introduction

This paper analyses non-linear strategies in a linear quadratic differential game (LQDG) played by two identical agents whose controls are bounded below by zero. Its theoretical motivation is a desire to address a concern dating back to Tsutsui and Mino (1990), who first treated non-linear strategies in this environment. Practically, the model is motivated by a greenhouse gas emissions problem.

A differential game is a game played in continuous time in which agents' choices cause a state variable to evolve according to a differential equation. The standard solution concept, the Markov perfect equilibrium (MPE), allows application of optimal control techniques to the game. Thus, differential games extend static games, repeated games and optimal control problems.

The workhorse of this literature has been the LQDG. Its name derives from the two equations defining a differential game, the state variable's equation of motion, and agents' instantaneous utility functions. In LQDG the former is a linear function of agents' controls and the state variable; the latter are quadratic in the same.[1] Not only do LQDG therefore seem to capture some of the spirit of many economic problems,[2] but they are known, under fairly mild conditions, to yield linear solutions, unique within the class of linear functions.

In the absence of further constraints, however, there is no reason to find these singular solutions more appealing than other ones. This observation has generated an interest in non-linear MPE in LQDG dating back to Tsutsui and Mino (1990). In their paper, duopolies chose output levels of a homogeneous good, causing a sticky price to evolve over time. In addition to the well known linear MPE, they found a continuum of implicitly defined non-linear ones. Economically, these were of particular interest when the steady state of a non-linear MPE was that of the first best – a version of the Folk Theorem for differential games.

Their finding has become the standard reference in papers on multiple equilibria and non-linear strategies in LQDG, and has been applied to a variety of settings, including environmental economics (q.v. Dockner and Long, 1993; Wirl and Dockner, 1995; Mäler et al., 2003), industrial organisation (q.v. Karp, 1996; Vencatachellum, 1998b) and the economics of the family (q.v. Feichtinger and Wirl, 1993).

At the same time, the paper generated disquiet by endogenising the domain over which strategies were defined, and their performance assessed. The differential equation derived in solving the Bellman's equation produced an infinite number of solutions. For each solution, the associated play lay in the interior of the action space only over a sub-domain of the state space. Tsutsui and Mino (1990) then evaluated play, and deviations, over these sub-domains.

[1] In an affine quadratic game, the equation of motion also includes a constant term. Both specifications will be referred to here as 'linear quadratic' games. This is less precise, but more concise.

[2] Fudenberg and Tirole (1991, 13.3.3) also note the hope that LQDG represent 'good Taylor approximation[s] to more general games'.

This endogenisation has serious implications: for a particular strategy to support a Nash equilibrium an agent must regard that strategy as yielding a superior payoff to any other admissible strategy – including those strategies specifying corner play, or causing the state variable to leave the endogenous sub-domain. By preventing their consideration, this approach ruled out the sort of calculation that underlies the Nash concept.

Possibly reflecting these concerns, a leading text on differential games does not include the non-linear solutions cited above (Başar and Olsder, 1999). In finite horizon affine quadratic games, without state or control bounds, the authors present the possibility of non-linear solutions as an 'unresolved problem' (Başar and Olsder, 1999, Remark 6.16). The results they present as the horizon becomes infinite are the limits of the finite horizon affine strategies.

This paper revisits the question of non-linear strategies. It does so in two steps. First, it presents a sufficiency result for the existence of MPE based on results in Dockner et al. (2000). The result is applicable well beyond LQDG. More specifically, it applies to infinite horizon differential games in which the value function may be unbounded below and may not be continuously differentiable.

Each of these latter two features introduces technical complications for sufficiency conditions. Generalised gradients are required at the non-differentiable points in the domain of a candidate value function. The larger problem, and that responsible for concerns about the existing literature, is the unbounded below value function. Over the infinite horizon, integrals associated with control paths may not converge. This, in turn, may require comparison of infinite payoffs. A standard solution to this has been to impose parameter constraints, or Uzawa conditions, to ensure finite valuation. The role of the endogenous sub-domains in Tsutsui and Mino (1990) and its successors is similar: a bounded domain bounds instantaneous utility; with impatience, this ensures finite valuation.

Rather than imposing parameter bounds, we follow a literature dating back to Ramsey (1928) which tries to handle infinite values directly. Rejecting discounting of utility as an 'ethically indefensible ... weakness of the imagination', Ramsey assumed satiation at finite levels of consumption. When actual consumption approached this level, the undiscounted series defined by the shortfall between instantaneous utility fell and 'bliss' was convergent.

To address situations without satiation, von Weizsäcker (1965) and Atsumi (1965) introduced what became known as the 'overtaking' criterion for comparing programmes with infinite value: a feasible programme is optimal under this criterion if its payoff stream (weakly) exceeds that of any other for all finite horizons beyond some finite T.[3]

As optimal programmes may not exist under this criterion, weaker criteria have also been introduced. Best known among these is the 'catching up' criterion of Gale

[3]Although introduced in the same journal issue, later writers have often unambiguously ascribed the criterion to one author or the other. I am grateful to Jim Mirrlees for suggesting that the criteria were independently defined.

(1967). This stands in the same relation to the overtaking criterion as an ε-equilibrium does to a standard equilibrium.[4]

As such criteria have not been widely applied to games, no consensus exists on their applicability. The sufficiency conditions presented by Dockner et al. (2000) use catching up optimality as their baseline criterion. As the sufficiency result presented here assembles a number of theirs, we also adopt this criterion.[5]

The paper's second step is to apply the sufficiency conditions to the solutions of the Bellman's equation generated by a standard LQDG. The specific model analysed here is closer to that in Dockner and Long (1993) in both form and motivation than it is to Tsutsui and Mino (1990). This has no analytical consequence: the techniques and results presented here are applicable not only to other LQDG, but to more general differential games. Outside of the LQDG framework, singular solutions need not be linear.

Unsurprisingly, we again explicitly derive the standard linear MPE. For the continuum of non-linear candidates, first reported by Tsutsui and Mino (1990), we present necessary and sufficient conditions for any of the non-linear candidates to be a MPE.[6] These conditions also apply to a piecewise linear strategy formed from the standard linear MPE and another singular solution to the differential equation. Although this strategy is a natural limit to the non-linear strategies, it has yet to be reported.

When non-linear MPE exist (including the piecewise linear MPE), their steady states are more conservative than that of the linear MPE, so may thus be closer to that of the first best. When the non-linear MPE do not exist, the steady state of the linear MPE is itself more conservative than that of the first best. Thus, underexploitation of the communal resource, as noted by Dutta and Sundaram (1993), may not be unusual. (Wirl (2005) argues that whether over or under-exploitation occurs depends on a condition on agents' elasticities of marginal utility in their own action.)

The condition for the existence of non-linear MPE loosens as agents become more patient; once patience exceeds a threshold, the continuum of non-linear MPE grows continuously. As this threshold depends on other model parameters, even perfect patience may not suffice to attain it. Coupled with the result that non-linear steady states are closer to that of the first best, this suggests a form of Folk Theorem for differential games, whereby patience allows attainment of the efficient outcome.

[4]Seierstad and Sydsæter (1977) and Dockner et al. (2000) review these criteria, and a weaker one yet, 'sporadically catching up'. Stern (1984) adds a further five criteria.

[5]As instantaneous utility is only unbounded below here, and as a negative infinite payoff is dominated by any finite one, these criteria may seem unnecessarily complicated. Their role becomes clearer when noting that either agent can set a control to ensure its opponent negative infinite payoffs. Under the standard criterion, any play by the opponent is weakly optimal. The standard criterion therefore admits as equilibria any pair of controls in which each agent ensures the other negative infinite payoffs.

[6]An earlier version of this paper (Rowat, 2000) and Rubio and Casino (2002) have both identified this condition, but neither substantiated it properly: Rowat (2000) did not recognise and address the problem of unbounded below value functions; Rubio and Casino (2002) did not require strategies to be defined over the whole of the state space.

Although multiple equilibria may arise, they are consistent with the unique optima found in the optimal growth literature with unbounded returns (q.v. Le Van and Morhaim, 2002, in discrete time): given any fixed play by the second agent, a single optimal control is derived for the first. Multiple equilibria thus result not from multiple best responses to given play, but to multiple (symmetric) fixed points.[7] As the lower bounds on the state space and on agents' controls goes to $-\infty$, the non-linear candidates are eliminated, leaving the standard linear MPE.

The linear quadratic model is presented in Section 2. Section 3 presents and solves its associated Hamilton–Jacobi–Bellman (HJB) equation. This produces a family of candidate MPE, which are assessed in Section 4. (Section 4.3 extends the results to generalised lower bounds.) Section 5 concludes. Appendix A presents the sufficiency conditions for equilibrium based on Dockner et al. (2000). These are weaker than required for our present application but should facilitate analysis of more general differential games. The definitions throughout also follow Dockner et al. (2000).

2. The linear quadratic model

Consider a symmetric, stationary differential game. There are two identical agents, $i \in \{1, 2\}$; refer to the agent other than i by $-i$. At each instant in time, t, each selects a control, $u^i(t)$, from its feasible set. With the play of the other, ϕ^{-i}, this influences the evolution of a state variable, x. Each seeks to maximise the present value, discounted at rate $r \in \Re_{++}$, of its utility stream.

The game thus outlined, $\Gamma(x_0, 0)$, may be formalised as

$$\max \quad J^i_{\phi^{-i}}(u^i(\cdot)) = \max \int_0^{\infty} \mathrm{e}^{-rt} F(x(t), u^i(t), \phi^{-i}(x(t)))\,\mathrm{d}t \tag{1}$$

$$\text{s.t.} \quad \dot{x} = f(x(t), u^i(t), \phi^{-i}(x(t))), \tag{2}$$

$$x(0) = x_0 \in X, \tag{3}$$

$$u^i(t) \in U(x(t), \phi^{-i}(x(t))). \tag{4}$$

The game is symmetric as: agents' instantaneous payoff functions and feasible sets take the same form; their ability to influence the state's evolution is identical.

It is stationary as the instantaneous payoffs, feasible sets and the equation of motion are not explicitly dependent on time. The second argument of $\Gamma(x_0, 0)$ refers to the time at which play begins. As stationary environments may admit non-stationary solutions, we retain the index to recognise this possibility.

The LQDG considered here restricts the above as follows:

$$F(x, u^i, \phi^{-i}) \equiv -(u^i - \xi)^2 - v(x - \zeta)^2, \tag{5}$$

$$f(x, u^i, \phi^{-i}) \equiv u^i + \phi^{-i} - \delta x, \tag{6}$$

[7] The 'technology' in the differential game cannot be classified ex ante as being of constant, increasing or decreasing returns, as in the optimal growth, as the transition process depends on both agents' play.

$$X \equiv \Re_{++},$$

$$U(x, \phi^{-i}) \equiv \Re_{+},$$

where δ, v, ξ and ζ are positive real constants.[8] The parameter restrictions imposed ensure that instantaneous utility is concave in both control and state.

An attractive property of LQDG is that, when the state and actions spaces are unbounded, mild conditions ensure the existence of equilibria in which the strategies are linear functions of the state variable alone, yielding value functions that are quadratic in the state. These are typically derived from a system of Riccati equations (Dockner et al., 2000, 7.1.3; Başar and Olsder, 1999, Proposition 6.8). When the strategy spaces are restricted to affine functions of the state variable, sufficient conditions for these solutions to be unique are known (Lockwood, 1996).

Under the commons problem interpretation, u^i may be thought of as nation i's greenhouse gas emissions, produced incidentally to national production (in a fixed ratio), x the atmospheric stock of greenhouse gasses and δ the decay or assimilation rate. Thus, agents have a production glut point ($u^i = \xi$) and a climate glut point ($x = \zeta$). The former may be consistent with an aggregated neo-classical labour supply trade off between work and leisure or an optimal capacity utilisation ratio. The latter allows agents to have some sense of optimal climate including, but not necessarily, the lunar climate, $\zeta = 0$.

2.1. Some reference payoffs

Two reference payoffs are presented to provide comparisons for payoffs arising from play of the game.

The payoff to being at the glut point, $(x, u^i) = (\zeta, \xi)$, forever is zero. While this is not attainable as a steady state except when $\delta\zeta = 2\xi$, it does impose a finite upper bound on payoffs. Thus, any solution to this problem must have a payoff that is bounded above by zero.

The steady state of the first best when agents are identical and equally weighted by the social planner may also be calculated. The current value Hamiltonian is

$$\mathscr{H} = -(u^1 - \xi)^2 - (u^2 - \xi)^2 - 2v(x - \zeta)^2 + m(u^1 + u^2 - \delta x),$$

where $m(t)$ is the current value Lagrangian multiplier, with first order conditions

$$m(t) = 2[u^i(t) - \xi],$$

$$\dot{m}(t) - (\delta + r)m(t) = 4v[x(t) - \zeta].$$

These imply a system of differential equations.[9] Rather than solving the full trajectories note that, in the steady state, $u^i(t) = (\delta/2)x(t)$ and $m(t) = -4v(x(t) - \zeta)/(\delta + r)$.

[8] The LQDG presented here is a special case: more general quadratic functions may include quadratic terms in u^{-i} and cross terms in u^i, ϕ^{-i} and x. Consideration of this simpler case merely facilitates expositional clarity.

[9] The dynamic programming approach does not give any clearer an expression for the dynamics. Its differential equation

Combined with the first order conditions these yield the steady state

$$(\bar{x}, \bar{u}^i) = \left(2\frac{(\delta + r)\xi + 2v\zeta}{\delta(\delta + r) + 4v}, \delta\frac{(\delta + r)\xi + 2v\zeta}{\delta(\delta + r) + 4v}\right). \tag{7}$$

When

$$\frac{\zeta}{\xi} \leqslant \frac{2}{\delta} \tag{A1}$$

the first best stock level exceeds the climate glut level; first best output falls below the product glut level. As this is the first best, though, it is optimal by definition and cannot be considered a ‘tragedy’ result.

As condition (A1) recurs throughout the paper we adopt it as an assumption in what follows. For now, we motivate the assumption on strictly expositional grounds, but it will be seen to determine whether the singular solution under or overprovides relative to the first best in steady state.

3. The Hamiltonian–Jacobi–Bellman equation

As the solution to equation of motion (6) depends on agents' play, restrictions on play are necessary to ensure a unique solution, $x(t)$. We impose regularity conditions to ensure this not out of concern that multiple solutions may preclude a maximum solution (q.v. Burton and Whyburn, 1952) but in order to allow agents to associate payoffs to their strategies.

Therefore, following the convention in Dockner et al. (2000) whereby $[0, T\rangle$ is read as $[0, \infty)$ when T is infinite and $[0, T]$ otherwise:

Definition 1 (*Dockner et al., 2000, 3.1*). A control path $u^i : [0, T\rangle \mapsto \Re$ is *feasible* for $\Gamma(x_0, 0)$ if the initial value problem defined by Eqs. (2) and (3) has a unique, absolutely continuous solution $x(\cdot)$ such that the constraints $x(t) \in X$ and $u^i(t) \in U(x(t), \phi^{-i})$ hold for all t and the integral in Eq. (1) is well defined.

Feasibility may also be referred to as admissibility (Dockner and Sorger, 1996). As in a generalised game (Debreu, 1952), the feasible set for agent i therefore depends on the actions taken by other agents. We shall see that feasible controls are consistent with multiple equilibria; each, however, induces a unique $x(t)$. See Başar and Olsder (1999, pp. 226–227) or Dockner et al. (2000, p. 40) for further discussion.

(*footnote continued*)

$$w'(x) = \frac{(\delta + r)w(x) + 4v(x - \zeta)}{2\xi - \delta x + w}$$

(where $w(x)$ is the derivative of the candidate value function, $W(x)$, and subscripts index agents) has an unwieldy implicit solution.

As the game is stationary, we focus on equilibria supported by stationary strategies. There may also be equilibria supported by non-stationary strategies.[10]

Definition 2 (*Dockner et al., 2000, p. 97*). A *stationary Markov strategy* is a mapping, $\phi^i : X \mapsto U^i$, so that the time path of the control $u^i(t) = \phi^i(x(t))$.

Thus, while the payoff-relevant state space may be very large, Markov strategies are functions of the current state alone.

Then:

Definition 3. A pair of functions $\phi^i : X \mapsto \Re, i \in \{1, 2\}$ is a *stationary Markov Nash equilibrium* if, for each $i \in \{1, 2\}$, an optimal control of problem 1 with constraints (2)–(4) exists and is given by the stationary Markov strategy $u^i(t) = \phi^i(x(t))$.

The following restricts the more general definition to stationary games:

Definition 4 (*Dockner et al., 2000, 4.4*). Let (ϕ^1, ϕ^2) be a Markov Nash equilibrium of $\Gamma(x_0, 0)$. The equilibrium is a MPE if, for each $(x, t) \in X \times [0, T\rangle$, the subgame $\Gamma(x, t)$ admits a Markov Nash equilibrium (ψ^1, ψ^2) such that $\psi^i(y, s) = \phi^i(y, s)$ for all $i \in \{1, 2\}$ and all $(y, s) \in X \times [t, T\rangle$.

When $\Gamma(x, t)$ is stationary, $\Gamma(x, 0) = \Gamma(x, t)$. Thus, all stationary Markov Nash equilibria are MPE (Dockner et al., 2000, p. 105).

Definition 5. Let the *value* of game $\Gamma(x_0, 0)$ to agent i be

$$V^i(x) = \max_{u^i \geqslant 0} J^i_{\phi^{-i}}(u^i(\cdot)).$$

The sufficiency conditions in Theorem 3 require that V^i be locally Lipschitz. By Rademacher's Theorem, Lipschitz continuous functions are almost everywhere differentiable (Clarke, 1983, p. 63). Tsutsui and Mino (1990) and Dockner and Long (1993) require the stronger assumption that $V^i(\cdot) \in \mathscr{C}^2$; this will be seen, in our environment, to follow automatically at most points for which our V^i are differentiable. Dockner and Sorger (1996) do not make continuity assumptions; instead, they derive MPE strategies which are discontinuous but which generate a continuous V^i. Başar and Olsder's Example 5.2 (Başar and Olsder, 1999) illustrates the same phenomena in single agent optimisation problems; the optimal control in their example follows a bang-bang pattern. When the value function is finite, Gota and Montruccio (1999) present sufficient conditions for the value function to be $\mathscr{C}^1$ with Lipschitz continuous derivative in spite of the optimal control only being interior for a short time interval.

When agent i's value function is differentiable it solves the HJB equation:

$$rV^i(x) = \max_{u^i \geqslant 0}[-(u^i - \xi)^2 - v(x - \zeta)^2 + V^i_x(x)(u^i + \phi^{-i} - \delta x)], \tag{8}$$

[10] See Dockner et al. (2000, Exercise 4.5) for an example.

given fixed play ϕ^{-i} by agent $-i$. By V_x^i we mean the derivative of $V^i(x)$; later it will refer to a partial derivative. As the equation of motion makes it impossible that $x(t) = 0$ if $x > 0$ no constraints are imposed on the state space in Eq. (8).

The non-negativity requirement on u^i provides a first order necessary condition for the optimal control:

$$u^{i*} \equiv \phi^{i*} \equiv \max\left\{0, \xi + \frac{V_x^i(x)}{2}\right\}. \tag{9}$$

As Eq. (8) is concave in its control, the optimal control is unique and a maximiser. Solutions to the HJB equation (8) are not, however, as they introduce a constant of integration.

Refer to situations in which $\phi^{i*} = 0$ as *corner solutions* and those in which $\phi^{i*} > 0$ as *interior solutions*. Call the inequality determining the greater term on the RHS of Eq. (9) the *auxiliary condition*.

3.1. The differential equation

Substitute the conditions of Eq. (9) into the HJB equation (8). As the differential equation generated produces a family of solutions, denote the family of *candidate value functions* so generated by $\mathscr{W}$; an individual member of that family is referred to as W. Therefore $V^i \in \mathscr{W}$. Substitute $\phi^{2*} = \phi^{1*}$ into the HJB equation to obtain

$$rW(x) = \left\{ \begin{array}{ll} -v(x-\zeta)^2 + W'(x)(2\xi - \delta x) + \dfrac{3W'(x)^2}{4}, & W'(x) \geqslant -2\xi \\ -\xi^2 - v(x-\zeta)^2 - \delta x W'(x), \quad W'(x) \leqslant -2\xi & \end{array} \right\}. \tag{10}$$

Symmetric play has now been imposed. The remainder of the analysis may be broken into two steps. The first, and standard, solves the two terms of Eq. (10); this occupies the next subsections. The more difficult and innovative step involves refining $\mathscr{W}$ to identify constants of integration consistent with the requirements of optimal play's value function.

3.1.1. Corner solutions

Eq. (10) is a linear ODE when $W'(x) \leqslant -2\xi$, and thus easily solved:

$$W(x) = -\frac{\xi^2 + v\zeta^2}{r} - \frac{v}{2\delta + r}x^2 + \frac{2v\zeta}{\delta + r}x + cx^{-r/\delta},$$

where c is a constant of integration. The condition on $W'(x)$ only allows this to hold for values of x satisfying

$$\frac{2\delta}{r}\left[\left(\xi + \frac{v\zeta}{\delta + r}\right)x^{(\delta+r)/\delta} - \frac{v}{2\delta + r}x^{(2\delta+r)/\delta}\right] \leqslant c. \tag{11}$$

As the exponent on Eq. (11)'s first term is smaller than that on the second, it dominates for small values of x. For larger x, though, the second term overpowers it. Fig. 1, a stylised plot of Eq. (11), illustrates the implications of this for solutions. For large values of c (e.g. c_2 in the figure) the condition for the corner solution is always

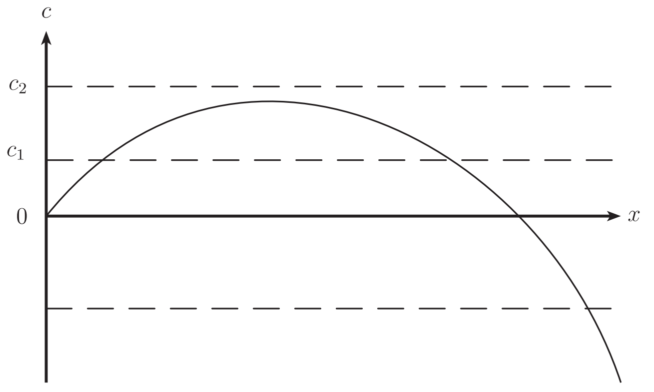

Fig. 1. Transitions between the corner and interior solutions.

satisfied and $\phi^{i*}(x) = 0$ is a solution to the HJB equation. For smaller values, e.g. c_1, it is satisfied for small x, is then violated, and finally is again satisfied for large values of x. For all $c \leqslant 0$ the condition is violated at small x but eventually comes to hold. It is expected that, when the corner solutions violate condition (11), the strategy will continue in the interior.

3.1.2. Interior solutions

The quadratic interior solution, Eq. (10) when $W'(x) \geqslant -2\xi$, is solved by differentiating it again.[11] The next lemma demonstrates when this is legitimate.

Lemma 1. *When* $W(x)$ *is defined by Eq.* (10) *and* $W'(x) \geqslant -2\xi, W(x) \in \mathscr{C}^\infty$ *if*

$$\tfrac{3}{2}W'(x) - \delta x + 2\xi \neq 0. \tag{12}$$

Proof. Rewrite Eq. (10) when $W'(x) \geqslant -2\xi$ as

$$W'(x) + \tfrac{2}{3}(2\xi - \delta x) = \pm\sqrt{\tfrac{4}{3}v(x-\zeta)^2 + \tfrac{4}{9}(2\xi - \delta x)^2 + \tfrac{4}{3}rW(x)}.$$

When this equation is not identically zero, each branch of the right-hand side is a $\mathscr{C}^\infty$ function of (W, x). In these cases, equivalent to condition (12), it follows from standard results on the smooth dependence of solutions with respect to initial conditions that $W(x) \in \mathscr{C}^\infty$ on its maximal interval of existence (q.v. Arnol'd, 1992, Section 7.3).[12] The multiplicity of branches has implications for uniqueness, not smoothness. □

Call the locus of points failing to satisfy inequality (12) the *non-invertibility* (NI) *locus*. The quadratic term in the interior component of Eq. (10) causes this to pass through the feasible state-action space. On the other hand, when condition (12) is not violated, the relevant portion of Eq. (10) may be differentiated. For notational

[11] This approach is also taken by Tsutsui and Mino (1990). Dockner and Sorger (1996) present a case in which direct integration is possible.

[12] I am grateful to an anonymous referee for suggesting this proof.

convenience define $w(x) \equiv W'(x)$. Therefore

$$w'(x) = \frac{(\delta + r)w(x) + 2v(x - \zeta)}{\frac{3}{2}w(x) - \delta x + 2\xi} \quad \text{when } x \geqslant 0. \tag{13}$$

The denominator cannot equal zero as that would violate the conditions of Lemma 1, preventing the differentiation performed to reach Eq. (13).

To solve Eq. (13) transform the equation into one that is homogeneous of degree zero in its variables by defining $\Omega \equiv w - a$ and $\Psi \equiv x - b$ to remove its constant terms. This requires that

$$a \equiv 2v\frac{\delta\zeta - 2\xi}{\delta(\delta + r) + 3v} \quad \text{and} \tag{14}$$

$$b \equiv \frac{2\xi(\delta + r) + 3v\zeta}{\delta(\delta + r) + 3v} > 0. \tag{15}$$

These definitions reduce the differential equation to

$$\frac{d\Omega}{d\Psi} = G\left(\frac{\Omega}{\Psi}\right) = \frac{(\delta + r)\Omega + 2v\Psi}{\frac{3}{2}\Omega - \delta\Psi} = \frac{(\delta + r)\Omega/\Psi + 2v}{\frac{3}{2}\Omega/\Psi - \delta}. \tag{16}$$

To exploit the homogeneity of Eq. (16) define $S \equiv \Omega/\Psi$. Therefore

$$[S^2 - \tfrac{2}{3}(2\delta + r)S - \tfrac{4}{3}v]\, d\Psi = (\tfrac{2}{3}\delta - S)\Psi\, dS,$$

which has two constant solutions,

$$S = \{s_a, s_b\} \equiv \tfrac{1}{3}\left[2\delta + r \pm \sqrt{(2\delta + r)^2 + 12v}\right], \tag{17}$$

with $s_a > 0 > s_b$. These are akin to the algebraic Riccati equations used to derive linear strategies.

Transforming these back into the original variables produces

$$\phi_a \equiv \xi + \tfrac{1}{2}[a + s_a(x - b)], \tag{18}$$

$$\phi_b \equiv \xi + \tfrac{1}{2}[a + s_b(x - b)]. \tag{19}$$

Defining x_b as the stock level at which the line $\phi_b(x)$ intersects the steady state locus (SSL), $\dot{x}(t) = 0$, yields

$$x_b \equiv \frac{2\xi + a - s_b b}{\delta - s_b}.$$

Therefore, by direct manipulation of the relevant definitions, including in Eqs. (7) and (15):

Lemma 2. (*Condition* (A1)) $\Leftrightarrow (a \leqslant 0) \Leftrightarrow (x_b \geqslant b \geqslant \bar{x} \geqslant \zeta)$.

When $S \notin \{s_a, s_b\}$, solve

$$\frac{d\Psi}{\Psi} = \frac{(\frac{2}{3}\delta - S)\, dS}{(S - s_a)(S - s_b)} = \frac{\gamma_a\, dS}{S - s_a} + \frac{\gamma_b\, dS}{S - s_b}, \tag{20}$$

when γ_a and γ_b are determined by the method of partial fractions to be

$$\gamma_a \equiv \frac{r}{3(s_b - s_a)} - \frac{1}{2} < 0 \quad \text{and} \quad \gamma_b \equiv \frac{-r}{3(s_b - s_a)} - \frac{1}{2} < 0,$$

so that $\gamma_a + \gamma_b = -1$. Integrating Eq. (20) when $S \notin \{s_a, s_b\}$ then yields

$$\ln|\Psi| = \hat{K} + \gamma_a \ln|S - s_a| + \gamma_b \ln|S - s_b|, \tag{21}$$

where $\hat{K}$ is a real constant of integration. Exponentiation produces

$$|\Psi| = \frac{1}{K}|S - s_a|^{\gamma_a}|S - s_b|^{\gamma_b}, \tag{22}$$

where $K \equiv \mathrm{e}^{-\hat{K}} \geqslant 0$. In terms of x and $W'(x)$ this becomes

$$K = |W'(x) - a - s_a(x - b)|^{\gamma_a}|W'(x) - a - s_b(x - b)|^{\gamma_b}.$$

Eq. (9) may be used to rewrite this in terms of $\phi^1(x)$ instead of $W'(x)$. Doing so does not change the form of the equation. The ϕ_a and ϕ_b solutions identified in Eqs. (18) and (19) correspond to $K = \infty$ (as γ_a and γ_b are negative); thus, each of these solutions sets one of the right-hand side terms to zero.

To sum up, the solution to differential equation (10) is

$$K = |W'(x) - a - s_a(x - b)|^{\gamma_a}|W'(x) - a - s_b(x - b)|^{\gamma_b} \tag{23}$$

when $W'(x) \geqslant -2\xi$ and condition (12) holds; and

$$W(x) = -\frac{\xi^2 + v\zeta^2}{r} - \frac{v}{2\delta + r}x^2 + \frac{2v\zeta}{\delta + r}x + cx^{-r/\delta} \tag{24}$$

when $W'(x) \leqslant -2\xi$. Eq. (23) is still undetermined: integration of $W'(x)$ will produce a second constant of integration. We shall see that K, the first constant of integration, indexes solutions while the second constant adjusts payoffs along given solution paths.

4. Candidate MPE

The relationship in Eq. (9) allows translation of Eqs. (23) and (24) into the space of Markov strategies, (x, ϕ^i). Three classes of solutions can then be seen to exist:

1. Corner solutions satisfying Eq. (24), so that $\phi^i(x) = 0$. Denote these by ϕ_0.
2. Singular solutions satisfying Eq. (23) when $K = \infty$. These are explicitly defined in Eqs. (18) and (19) and denoted by ϕ_a and ϕ_b, respectively. These intersect at $(x, \phi^i) = (b, \xi + \frac{1}{2}a) > \mathbf{0}$, inside the feasible (x, ϕ^i) space. Further, when (A1) holds, their intersection is above the climate glut point ($b > \zeta$) and below the product glut point ($\xi + \frac{1}{2}a < \xi$). As there is a non-unique solution to the differential equation at this intersection, call that point a *singularity* and the strategies passing through it *singular solutions*.
3. Non-linear solutions satisfying Eq. (23) when K is finite. As K is arbitrary, Eq. (23) describes a family of infinitely many solutions. Each of these is a member

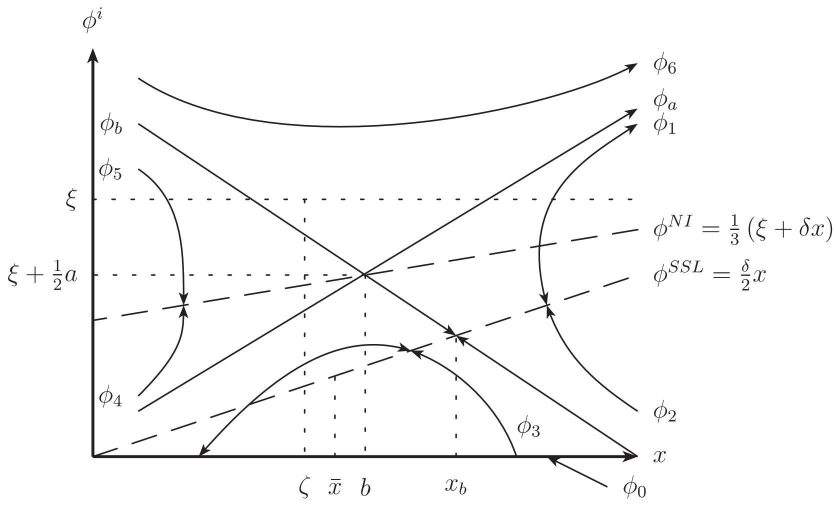

Fig. 2. Markov strategies when (A1) holds.

of one of six types of solutions, representatives of which are displayed in Fig. 2; these are denoted by $\phi_1, \ldots, \phi_6$.Thus, the ϕ_3 solutions are bounded above by the singular solutions, and the ϕ_6 solutions bounded below. The rationale for distinguishing between ϕ_1 and ϕ_2 (or ϕ_4 and ϕ_5) shall become apparent.

In all cases, the arrows in Fig. 2 indicate the variables' evolution in time. It also displays the SSL, defined by $\mathrm{d}x/\mathrm{d}t = 0$, and the NI locus, along which $\mathrm{d}\phi^i/\mathrm{d}x = \pm\infty$. In the present case, these are

$$\phi^{\mathrm{SSL}} = \frac{\delta}{2}x \quad \text{and} \quad \phi^{\mathrm{NI}} = \frac{\delta}{3}x + \frac{\xi}{3},$$

respectively.

Fig. 2 is equivalent to Fig. 1 in Tsutsui and Mino (1990).[13] The diagrams are oriented differently as Tsutsui and Mino's diagram presents the transformed control variable along the vertical axis while that here presents the control variable itself. Thus, their upper bound, $y = (p - c)/s$, is the present $\phi^i = 0$.

By the HJB equation's first order condition, Eq. (9), $W'(x) < 0$ when $\phi^i < \xi$, implying that increases in the initial stock, x, always reduce the value of the game when agent i plays at less than the product glut level. This may seem particularly surprising when $x < \zeta$ and above the SSL as x increases in time towards the climate glut point. This benefit is apparently balanced by a loss in the product term and, in some cases, a moving more quickly in time beyond the climate glut point.

[13] Cf. also Fig. 1 in Dockner and Long (1993) and in Vencatachellum (1998a). Use Lemma 2 to redraw Fig. 2 when assumption (A1) is violated as follows: ϕ_a and ϕ_b intersect below the SSL; as ϕ_a is steeper than the SSL, $\phi_a(0) < 0$.

Candidate strategies must be able to map from any element of the state space, X. The ϕ_6 family of strategies, ϕ_0 and ϕ_a (when it does not intersect the horizontal axis) already do so. The interior solutions ϕ_b, ϕ_3 and ϕ_a (when it does intersect the horizontal axis) are extended by ϕ_0 when they trigger the auxiliary condition, $W'(x) = -2\xi$; denote these extensions by a caret so that $\hat{\phi}_p \equiv \max\{0, \phi_p\}$, where p indexes solution families.

Finally, although the candidate strategies will generally be kinked at the corner extension, this is consistent with the requirement that V^i be locally Lipschitz.

4.1. Refining the candidate strategy set

A solution to differential equation (10), $W(x)$, is still two steps removed from describing payoffs under MPE play. First, it must be demonstrated that $W(x) = V^i(x)$, that the candidate value function is a value function. We refine the candidate set against this requirement with two tests: do they define functions; are those functions bounded above? Second, we test the remaining candidates against the sufficiency conditions of Theorem 3 in Appendix A.

Lemma 3. *Members of the ϕ_1, ϕ_2, ϕ_4 and ϕ_5 families of solutions to differential equation* (10) *cannot form candidate MPE strategies.*

Proof. When members of the ϕ_1, ϕ_2, ϕ_4 and ϕ_5 solution families intersect the NI locus they cease to be functions. If they are to remain under consideration, some extension to them must be made so that they remain functions over X. They cannot be extended by ϕ_0 as, when they cease to be functions in X, they do not satisfy the auxiliary condition on $W'(x)$. No other extensions are possible. □

It is tempting to consider jumps from one of these solutions to, say, ϕ_0. However, no strategy constructed with jumps like this solves differential equation (10). Now consider $\hat{\phi}_{ab} \equiv \max\{0, \min\{\phi_a, \phi_b\}\}$, created by switching from ϕ_a to ϕ_b at their intersection.[14] This is not one of the trivial solutions presented in Eq. (17) but it is an interior solution to ODE (10). That it is not differentiable at $x = b$ does not disqualify it either: Lemma 1 only applies off the NI locus. The candidate $\hat{\phi}_{ba} = \phi_{ba} \equiv \max\{\phi_a, \phi_b\}$ may be discarded:

Lemma 4. *$W(x) \neq V^i(x)$ along $\hat{\phi}_a, \hat{\phi}_{ba}$ and the ϕ_6 family of strategies.*

Proof. Along $\hat{\phi}_a : x \to \infty \Rightarrow W'(x) \to \infty \Rightarrow W(x) \to \infty$, an impossible integral of the bounded above instantaneous utility function (5); this applies to $\hat{\phi}_{ba}$ as well. As $\phi_6(x) > \hat{\phi}_a(x)$, the ϕ_6 family produces the same contradiction. □

The argument that $\hat{\phi}_a, \hat{\phi}_{ba}$ and the $\hat{\phi}_6$ family do not provide candidate MPE strategies may be illustrated by demonstrating a profitable deviation from their play: as $s_a > \delta$, there is an x such that $\dot{x} > 0$ and $\phi^i(x) > \xi$ for all greater values of x along these strategies. An agent can then improve its payoff by capping play at $\phi^i = \xi$;

[14] I am grateful to an anonymous referee for encouraging consideration of this candidate.

doing so sets the utility loss term in production to zero and slows the climate loss term's growth (as compared to playing $\phi^i > \xi$).

Discarding $\hat{\phi}_a, \hat{\phi}_{ba}$ and the ϕ_6 family leaves only $\phi_0, \hat{\phi}_b, \hat{\phi}_{ab}$ and the $\hat{\phi}_3$ family of strategies to consider as possible MPE strategies.

Lemma 5. *$W(x) \neq V^i(x)$ in any candidate that satisfies $\phi^i(0) = 0$ and possesses constant of integration $c \neq 0$ in that cornered component.*

Proof. By Eq. (24)

$$\lim_{x \to 0} W(x) = -\frac{\xi + v\zeta^2}{r} + c \lim_{x \to 0} x^{-r/\delta}.$$

When $c > 0$, this unbounded limit again contradicts the bounded above instantaneous utility function. As noted in Eq. (11), which provided the condition for the solution to differential equation (10) to remain in the corner, $c < 0$ and $\phi^i(0) = 0$ are contradictory. □

The candidate with $c = 0$ is not eliminated by Lemma 5; it leaves $\phi^i(0) = 0$ immediately.

Again, express this rejection of $\phi^i(0) = 0$ in terms of profitable deviations by considering play at $x < \zeta$, the glut climate. The cornered strategy requires that agent 1 accept a climate loss as x continues to fall; defection to some small $\phi^i > 0$ reduces the climate loss and provides a production gain.[15]

Therefore:

Lemma 6. *When*

$$a - s_a b < -2\xi, \tag{25}$$

$\hat{\phi}_{ab}$ and the $\hat{\phi}_3$ candidate strategies are eliminated.

Proof. As $\hat{\phi}_{ab}$ and the $\hat{\phi}_3$ strategies are bounded above by $\hat{\phi}_a$, Lemma 5 rules them out when $\phi_a(0) < 0$; this is equivalent to condition (25). □

Condition (25) parallels that discussed in Rubio and Casino (2002, Section 4).[16] It holds for all v sufficiently small. In the extreme, when $v = 0$, the remaining singular

[15]Similar reasoning would also apply to an $\hat{\phi}_3$ member for which $\phi^i(0) > 0$ but which then declined to $\phi^i(x) = 0$ at some $0 < x < \zeta$. Eq. (11) reveals that this is an impossibility: the $\hat{\phi}_3$ path that passes through $(0, 0)$, and therefore attains $\phi^i(x) = 0$ at the lowest x, is identified by $c = 0$ along its corner component. This constant sets $\phi^i(x) = 0$ at

$$x \in \left\{0, \frac{2\delta + r}{v}\left(\xi + \frac{v\zeta}{\delta + r}\right)\right\}.$$

As this second value exceeds ζ for non-negative parameters, the impossibility is established.

[16]The equivalent condition in the environment and notation of Tsutsui and Mino (1990) is

$$\beta - \alpha z_a \geqslant -\frac{c}{s}.$$

Thus, very sticky prices, $s \to 0$, remove the non-linear candidates from the MPE set.

candidate reduces to $\hat{\phi}_b(x) = \xi$: without a stock effect, there is no interaction between the agents; they statically optimise with respect to production.[17]

Finally, the following lemma links inequality (25) with the paper's main inequality:

Lemma 7. *Assumption* (A1) *is necessary for* $\hat{\phi}_{ab}$ *or a* $\hat{\phi}_3$ *candidate to be an equilibrium strategy.*

Proof. By Lemma 6, inequality (25) suffices to eliminate $\hat{\phi}_{ab}$ and the $\hat{\phi}_3$ candidates. Thus, the complementary inequality,

$$\frac{\zeta}{\xi} < \frac{2}{3} \frac{3v + (\delta + r)\left[\delta - r - \sqrt{(2\delta + r)^2 + 12v}\right]}{v\left[r + \sqrt{(2\delta + r)^2 + 12v}\right]}, \tag{26}$$

is necessary for their survival. Direct comparison shows that the right-hand side of inequality (A1) exceeds that of inequality (26):

$$\sqrt{(2\delta + r)^2 + 12v} \geqslant \delta - r.$$

As (A1) is more relaxed than inequality (26), it is necessary for inequality (26), establishing the result. □

The complementary condition, (26), holds whenever the climate glut, ζ, is sufficiently small relative to the product glut, ξ. As the derivative with respect to r of its right-hand side is negative, it also holds when agents are sufficiently patient.[18] The existence of $\hat{\phi}_3$ candidates whose steady states lie closer to $\bar{x}$ than does that of $\hat{\phi}_b$ when condition A1 holds (q.v. Fig. 2) motivated a search for 'Folk Theorem' results whereby the efficient solution could be obtained by sufficiently patient agents (Tsutsui and Mino, 1990, p. 154; Dockner and Long, 1993). When agents are sufficiently impatient, this set of more efficient candidates is eliminated.

4.2. *Equilibrium*

Having discarded various families of solutions from further consideration, we now establish the main result, proving that certain candidates do support MPE.

Theorem 1. $\hat{\phi}_b$ *and any* $\{\hat{\phi}_p | \phi_p(0) \geqslant 0, p \in \{3, ab\}\}$ *are MPE strategies.*

The proof applies the sufficiency conditions in Theorem 3 (Appendix A). Accordingly, its structure and conditions will be more easily understood after reading Theorem 3.

Proof. 1. As the derivatives of $\hat{\phi}_b, \hat{\phi}_{ab}$ and the $\hat{\phi}_3$ are bounded, when they exist, the candidate Markov strategies are Lipschitz continuous. This, by the Picard–Lindelöf

[17] Dockner and Long (1993, p. 23) also note this result.
[18] When $\delta^2 > v$, perfect patience ensures that the condition holds.

Theorem, suffices for the initial value problem defined in Eq. (6) when $u^i(t) = \phi^i(x(t))$ to admit a unique solution (Walter, 1998, Section 6). As this solution may be expressed as the integral

$$x(t) = e^{-\delta t}\left\{x_0 + \int_0^t e^{\delta s}[\phi^i(x(s)) + \phi^{-i}(x(s))]\,ds\right\}, \tag{27}$$

it is absolutely continuous (Royden, 1988). A pair of these candidates is therefore feasible for $\Gamma(x_0, 0)$.

2. (a) Now consider non-stationary value functions and HJB equations. When $V^i(x, t; T)$ is differentiable, it must solve:

$$rV^i(x,t;T) = \left\{\begin{array}{l} -v(x-\zeta)^2 + V^i_x(x,t;T)(2\xi - \delta x) + \dfrac{3V^i_x(x,t;T)^2}{4} \\ \quad +V^i_t(x,t;T) \text{ when } V^i_x(x,t;T) \geqslant -2\xi; \text{ and} \\ -\xi^2 - v(x-\zeta)^2 - \delta x V^i_x(x,t;T) + V^i_t(x,t;T) \\ \quad \text{when } V^i_x(x,t;T) \leqslant -2\xi \end{array}\right\} \tag{28}$$

as

$$u^{i*} = \hat{\phi}^{i*} = \max\left\{0, \xi + \frac{V^i_x(x,t;T)}{2}\right\}.$$

This is solved by separation of variables, decomposing the value function into a stationary and a non-stationary component:

$$V^i(x,t;T) = W(x) - e^{r(t-T)}W(X(T;x,t)), \tag{29}$$

where play begins in subgame $\Gamma(x,t)$; $X(T;x,t)$ is the state reached at time T by candidate play and equation of motion (6) when $u^i(t) = \phi^i(x(t))$; $W(x)$ corresponds to the candidate solution, and is subject to the same auxiliary condition. Function (29) inherits continuity from $W(x)$.

To see that Eq. (29) solves the HJB equations, let $X_n(T;x,t)$ be the partial derivative of $X(T;x,t)$ with respect to its nth argument. Then substitute Eq. (29) and its partial derivatives,

$$V^i_x(x,t;T) = W'(x) - e^{r(t-T)}W'(X(T;x,t))X_2(T;x,t),$$

$$V^i_t(x,t;T) = -e^{r(t-T)}[rW(X(T;x,t)) + W'(X(T;x,t))X_3(T;x,t)],$$

into the general HJB equation for

$$\begin{aligned} r[W(x) - e^{r(t-T)}W(X(T;x,t))] = {} & F(x,\phi^i(x),\phi^{-i}(x)) \\ & + [W'(x) - e^{r(t-T)}W'(X(T;x,t))X_2(T;x,t)] \\ & \times f(x,\phi^i(x),\phi^{-i}(x)) - e^{r(t-T)}[rW(X(T;x,t)) \\ & + W'(X(T;x,t))X_3(T;x,t)]. \end{aligned} \tag{30}$$

As

$$rW(x) = F(x, \phi^i(x), \phi^{-i}(x)) + W'(x)f(x, \phi^i(x), \phi^{-i}(x)),$$

Eq. (30) may be simplified to

$$0 = e^{r(t-T)}W'(X(T;x,t))[X_2(T;x,t)f(x, \phi^i(x), \phi^{-i}(x)) + X_3(T;x,t)]. \tag{31}$$

Assessing this equation requires expressions for $X_2(T;x,t)$ and $X_3(T;x,t)$.[19] Letting $h(x) \equiv f(x, \phi^i(x), \phi^{-i}(x))$, the former may be derived from the equation of motion

$$X_1(s;x,t) = h(X(s;x,t)), X(t;x,t) = x,$$

by differentiating with respect to x and then integrating by separation of variables over $s \in [t, T]$ for

$$X_2(T;x,t) = \exp\left[\int_t^T h'(X(s;x,t))\,\mathrm{d}s\right].$$

Doing the same with t instead of x yields

$$X_3(T;x,t) = -h(x)X_2(T;x,t),$$

as the differentiated initial condition sets $-X_3(t;x,t) = X_1(t;x,t) = h(x)$. Substitution into Eq. (31) shows that the square bracketed term is identically zero.

Finally, Eq. (29) satisfies terminal condition (34): $V^i(x,T;T) = W(x) - e^{r(T-T)}W(x) = 0$ as $X(T;x,t) = x$ when $t = T$.

(b) As $\hat{\phi}_b, \hat{\phi}_{ab}$ and the $\hat{\phi}_3$ candidate strategies are derived from HJB equations, they are elements of $\Phi(x,t;T)$. The $\hat{\phi}_3$ candidate strategies and $\hat{\phi}_b$ are differentiable except when they corner; $\hat{\phi}_{ab}$ is non-differentiable when it corners and at $x = b$. In all cases, the Lebesque measure of times at which this occurs is zero.

3. Limit condition (37) is satisfied as

$$\lim_{T\to\infty} V^i(x,t;T) = W(x) - \lim_{T\to\infty} e^{rt}e^{-rT}W(X(T;x,t)) = W(x),$$

as $X(T;x,t)$ along all candidates are bounded are $T \to \infty$.

The limits are also finite as the candidate $W(x)$ are. Along a corner component, this is immediate from inspection of Eq. (24) as $x>0$. Along an interior component, the undetermined constant of integration needed to uniquely identify Eq. (23) is supplied by the discounted intertemporal integral on the right-hand side of expression (1). For any finite x, the candidate controls are also finite, so that instantaneous utility and its discounted infinite integral are as well. The limit value functions are not just locally Lipschitz but $\mathscr{C}^1$ as the candidate controls are continuous. Finally, the candidates were generated by solving the limit HJB equation (8).

4. For T sufficiently large, play along the candidates is described by Eq. (23). As $W'(x(T))$ is finite, so is $W(x(T))$ if the corresponding constant of integration is finite. The constant may be seen to be finite by integrating backwards from the candidate's stable steady state, which yields finite instantaneous utility. □

[19] I am grateful for the Associate Editor's suggestions in deriving these.

The candidate strategies eliminated by Lemmata 3–6 do not satisfy the sufficiency conditions of Theorem 3: those eliminated by Lemma 3 violate the Theorem's condition (1) by failing to provide a well-defined problem over the whole domain; those eliminated by the remaining Lemmata violate finite limit condition (3).

4.3. *Equilibrium with generalised lower bounds*

The preceding analysis allows generalised lower control and state bounds to be considered. Suppose now that $X \equiv \langle \underline{x}, \infty)$ and $U(\cdot) \equiv \langle \underline{u}, \infty)$ where $\underline{u}$ and $\underline{x}$ are negative.

First order condition (9) then becomes

$$u^{i*} \equiv \phi^{i*}(x) \equiv \max\left\{\underline{u}, \xi + \frac{V_x^i(x)}{2}\right\},$$

so that the corner component of linear differential equation (10) is now

$$rW(x) = -(\underline{u} - \xi)^2 - v(x-\zeta)^2 + (2\underline{u} - \delta x)W'(x),$$

when $W'(x) \leqslant -2(\xi - \underline{u})$. Its solution, the generalisation of Eq. (24), is

$$\begin{aligned} W(x) = & -\frac{\xi^2 + v\zeta^2 + (\underline{u} - 2\xi)\underline{u}}{r} - \frac{vx^2}{2\delta + r} + \frac{2v\zeta x}{\delta + r} + c\left(x - \frac{2\underline{u}}{\delta}\right)^{-r/\delta} \\ & - 4v\underline{u}[(x-\zeta)r + 2(\underline{u} - \delta\zeta)], \end{aligned} \tag{32}$$

subject to the same auxiliary condition. Eq. (23) is unaltered.

To generalise Lemma 6 first define

$$x_a \equiv \frac{2\xi + a - s_a b}{\delta - s_a}.$$

As x_b occurs at the intersection of the SSL and the line ϕ_b, x_a lies on the intersection of u_a and the SSL. Then:

Lemma 8. *When $x_a > \underline{x}$ and $\phi_a(x_a) > \underline{u}$, $\hat{\phi}_{ab}$ and the $\hat{\phi}_3$ candidate strategies are eliminated.*

Proof. The first condition generalises inequality (25). Together, the conditions ensure that $\hat{\phi}_{ab}$ and the SSL intersect at an unstable steady state. There are then two possibilities:

1. $2\underline{u}/\delta \in \langle \underline{x}, x_a)$ so that $\hat{\phi}_{ab}$ and the SSL intersect again at $(x, \phi) = (2\underline{u}/\delta, \underline{u})$. Generalising the proof of Lemma 5 takes the limit of Eq. (32) as $x \searrow 2\underline{u}/\delta$. The analogous condition to inequality (11) is

$$\frac{2\delta}{r}\left(x - \frac{2\underline{u}}{\delta}\right)^{(\delta+r)/\delta}\left[\xi + \frac{v\zeta}{\delta + r} - (1 + 2vr)\underline{u} - \frac{v}{2\delta + r}x\right] \leqslant c.$$

As $\underline{u} \leqslant 0$, $c < 0$ is ruled out, so that the limit produces $W(x) \to \infty$.

2. $2\underline{u}/\delta \notin \langle \underline{x}, x_a)$ so that $\hat{\phi}_{ab}$ and the SSL do not intersect for any $x < x_a$. Thus, $\forall x \in \langle \underline{x}, x_a), (u^i(t) - \xi)^2$ and $(x(t) - \zeta)^2$ increase over time along the candidate. Deviating to a $u^i \leqslant \xi$ above the candidate play that increases $\dot{x}(t) \leqslant 0$ reduces an agent's production loss and slows the growth of its climate loss, a profitable deviation. $\square$

The candidate strategies rejected by Lemma 8 correspond to candidate value functions that are generically discontinuous at the unstable steady state, violating sufficiency condition (3) of Theorem 3. This may be explicitly shown with $\hat{\phi}_{ab}$, whose piecewise linear structure produces a quadratic candidate value function.

Lemma 8 allows consideration of the textbook case (Başar and Olsder (1999, Section 7.1) and Dockner et al. (2000, Section 6.5.3)), in which the state and control space is unbounded: $\underline{u}, \underline{x} \to -\infty$, eliminating $\hat{\phi}_{ab}$ and the $\hat{\phi}_3$ candidates.

Finally, we generalise Theorem 1:

Theorem 2. *$\hat{\phi}_b$ and any $\{\hat{\phi}_p | \phi_p(2\underline{u}/\delta) \geqslant \underline{u}, p \in \{3, ab\}\}$ are MPE strategies.*

The proof is straightforward. The condition ensures that the candidate strategy does not intersect the SSL at an unstable steady state or along its corner component. The unique intersection between the candidate $\hat{\phi}_{ab}(x)$ and the SSL is at x_b, the stable steady state.

5. Discussion

The MPE presented above all possess steady states that are unique, stable and finite in (x, ϕ^i) space. The presence of an unstable steady state generically gives rise to discontinuities in the value function. While these therefore fail to satisfy this paper's sufficiency conditions, it is the existence of profitable deviations that eliminates them.

At the same time, the possession of a unique, stable and finite steady state is not sufficient for equilibrium: the ϕ_0 candidate satisfies these criteria, but generates an explosive candidate value function. This possibility should therefore be considered (by examining the equivalents of Eqs. (23) and (24)) when analysing differential games other than the linear-quadratic.

Kinks in agents' controls are also consistent with equilibrium play, both in the transition between corner and interior play and between singular elements of interior play.[20] This may not be the case in differential games in which a finite measure of time is spent at a kink.

We conclude by mentioning some avenues for future research. First, the techniques presented here do not depend on the linear quadratic structure of the game. The primary role of that structure is to reduce the singular MPE to a linear MPE. This simplifies derivation of the singular solution, allowing an explicit solution, but does not otherwise bear on the existence of non-singular solutions. Even without explicit solutions, the techniques presented here often rely on limits as

[20] As noted earlier, discontinuous controls may also support equilibrium play.

x tends to some value, which can also be applied to implicit solutions. Thus, we hope that the results presented here ease analysis of a larger class of differential games.

Second, additional or more complicated control bounds could be introduced without new techniques. Tsutsui (1996) considers capacity constraints on firms' production decisions. Analysis of non-constant upper bounds (e.g. a joint savings account game in which maximum aggregate withdrawal is the account balance) may be an interesting generalisation.

Finally, most existing analyses have been symmetric: agents are assumed to be identical; the search for MPE has been confined to ones in which they play identically. Technically, this reflects the possibility that a system of ODEs, one for each player, may not yield an analytical solution. Rowat (2002) uses numerical techniques to analyse the game presented here, first allowing identical agents to play differently, then allowing non-identical agents.

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Appendix A. Sufficiency conditions

The appendix presents a sufficiency theorem for the existence of Markov Nash equilibria based on results in Dockner et al. (2000). (It also applies to MPE in the stationary game $\Gamma(x, 0)$: the time index of any subgame can be re-set to zero.) This result then allows the conclusions of Theorem 1. We present the theorem in greater generality than necessary for Theorem 1 to increase its applicability to other environments. In particular, it is not limited to LQDGs.

The theorem is based on Theorem 4.1 of Dockner et al. (2000), a sufficiency condition for Markov Nash equilibria. The statement is generalised by their Theorems 3.4 and 3.5 to allow for value functions that are unbounded below and not continuously differentiable, respectively. The proof of the theorem is therefore left to Dockner et al. (2000).

Before proceeding, we present two further definitions necessary to state the result.

Definition 6 (*Dockner et al., 2000, 3.2*). Define $J^i_{\phi^{-i},T}$ by replacing the upper limit of integration in objective functional 1 with T. A feasible control path $u^i(\cdot)$ is *catching up optimal* if, for every other feasible control path $\tilde{u}^i(\cdot)$:

$$\liminf_{T\to\infty} [J^i_{\phi^{-i},T}(u^i(\cdot)) - J^i_{\phi^{-i},T}(\tilde{u}^i(\cdot))] \geqslant 0.$$

As noted by Gale (1967), which introduced the criterion,[21] this is equivalent to: for every $\varepsilon > 0$ there exists a $\bar{T}$ such that, for all $T \geqslant \bar{T}$,

$$[J^i_{\phi^{-i},T}(u^i(\cdot)) - J^i_{\phi^{-i},T}(\tilde{u}^i(\cdot))] \geqslant -\varepsilon.$$

Thus, equilibria in catching up optimal strategies are ε-equilibria for all sufficiently large T. The more stringent criterion of overtaking optimality sets $\varepsilon = 0$.

Stern (1984) presents seven other definitions of optimality in the infinite horizon framework.

To address the possibility of the non-differentiability of V, we also define the generalised gradient, or Clarkian:

Definition 7 (*Clarke, 1983, 2.5.1; Dockner et al., 2000, 3.4*). Let $V : \Re^2 \mapsto \Re$ be Lipschitz continuous in an open neighbourhood of x. The *generalised gradient* of V at x is the set

$$\partial V(x) = co\left\{ \lim_{i \to \infty} \nabla V(x_i) | x_i \to x, x_i \notin Z_V \right\},$$

where Z_V is the set of non-differentiable points of V.

Theorem 3. *Consider $\Gamma(x_0, 0)$, as defined above. Let (ϕ^1, ϕ^2) be a given pair of functions $\phi^i : X \mapsto \Re$ and assume that*:

1. *the pair (ϕ^1, ϕ^2) is feasible for $\Gamma(x_0, 0)$*;
2. *for all sufficiently large $T > 0$, and $i \in \{1, 2\}$*:

(a) *there exist locally Lipschitz continuous functions, $V^i(\cdot, \cdot; T) : X \times [0, T] \mapsto \Re$ which solve the HJB equations*

$$rV^i(x, t; T) = \max\{F(x, u^i, \phi^{-i}) + \alpha^i f(x, u^i, \phi^{-i}) + \beta^i \\ |u^i \in U(x, \phi^{-i}), (\alpha^i, \beta^i) \in \partial V^i(x, t; T)\} \quad \forall (x, t) \in X \times [0, T], \quad (33)$$

and the terminal condition

$$V^i(x, T; T) = 0. \quad (34)$$

Denote by $\Phi^i(x, t; T)$ the set of all $(u^i, \alpha^i, \beta^i) \in U(x, \phi^{-i}) \times \partial V^i(x, t; T)$ which maximise the RHS of HJB equations (33);

(b) *for the feasible control u^i_T against ϕ^{-i}, which induces state trajectory x_T, there exist $(\alpha^i(t), \beta^i(t)) \in \Re^2$ for all t such that*

$$(u^i_T(t), \alpha^i(t), \beta^i(t)) \in \Phi^i(x_T, t; T) \quad \textit{and} \quad (35)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V^i(x_T(t), t; T) = \alpha^i(t)\dot{x}_T(t) + \beta^i(t), \quad (36)$$

for almost all $t \in [0, T]$;

3. *for all $x \in X$ and $i \in \{1, 2\}$, the limits*

$$V^i(x) \equiv \lim_{T \to \infty} V^i(x, t; T) \quad (37)$$

[21] Carlson et al. (1991) refer to it as 'overtaking optimal'.

exist, are finite, and are locally Lipschitz continuous functions $V^i : X \mapsto \Re$ *which solve the HJB equations*

$$rV^i(x) = \max\{F(x, u^i, \phi^{-i}) + \alpha^i f(x, u^i, \phi^{-i}) \\ |u^i \in U(x, \phi^{-i}), \alpha^i \in \partial V^i(x)\} \quad \forall x \in X. \tag{38}$$

Denote by $\Phi^i(x)$ *the set of all* $(u^i, \alpha^i) \in U(x, \phi^{-i}) \times \partial V^i(x)$ *which maximise the RHS of HJB equations* (38);

4. $\limsup_{T\to\infty} e^{-rT} V^i(x(T)) \leqslant 0 \quad \forall i \in \{1,2\}$.

If $\phi^i(x) \in \Phi^i(x)$ *for each* $i \in \{1,2\}$ *and almost all* $t \in [0,\infty)$, *then* (ϕ^1, ϕ^2) *is a MPE in the sense of catching up optimality.*

Informally, assumption 1 ensures that each agent faces a well-defined problem: given play ϕ^{-i} by the other, the objective functional of agent i is well-defined – although it may take on infinite value.

Assumption 2a establishes sufficiency conditions for MPE in the finite horizon problem. Assumption 2b addresses the possibility of the failure of $V^i(\cdot;T)$ to be differentiable; by Rademacher's Theorem, locally Lipschitz continuous functions are differentiable for almost all $t \in [0,T]$ (Clarke, 1983, p. 63). By Theorems 3.1 and 3.5 in Dockner et al. (2000), these assumptions define an optimal control, $u^i_T(\cdot)$ over $t \in [0,T]$.

Assumption 4 is a transversality condition. It may be replaced with other conditions to obtain other criteria of optimality mentioned in the introduction. Finally, assumption 3 allows application of the transversality conditions to the sufficiency conditions derived under a finite horizon. If the optimal controls derived have a Markov representation, then a MPE exists.

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