2023 - March, 2nd

OPEN-LOOP NASH EQUILIBRIUM (OLNE)

Alessandra Buratto

.

DIPARTIMENTO MATEMATICA

Università degli Studi di Padova

(Dockner p.87)

$$
\begin{aligned}
\max_{u\geq 0} J^1(u(\cdot)) &= \int_0^T e^{-rt} \overbrace{\left[v(t) - x(t) - \frac{\alpha}{2}u^2(t)\right]}^{F^1} dt \\
\max_{v\in[0,1]} J^2(v(\cdot)) &= \int_0^T e^{-rt} [v(t) - x(t)]\, dt \\
\text{s.t.} \quad \dot{x}(t) &= 1 + v(t) - u(t)\sqrt{x(t)}, \\
x(0) &= x^0
\end{aligned}
$$

Assume $v(t) = \psi(t)$ for P2, find best response strategy for P1 $(P_0 = 1)$

$H^{C1}(x, u, \lambda, t) = P_0(\psi - x - \frac{\alpha}{2}u^2) + \lambda(1 + \psi - u\sqrt{x})$

Assume $u(t) = \Phi(t)$ for P1, find best response strategy for P2

$H^{C2}(x, v, \lambda, t) = P_0(v - x) + \mu(1 + v - \Phi\sqrt{x})$ $(P_0 = 1)$

$$H^{c1}(x,u,\lambda,t) = 0 - x - \frac{\alpha}{2}u^2 + \lambda\left(1 + s - u\sqrt{x}\right)$$

- $u^* = \underset{u \geq 0}{\operatorname{argmax}}\ H^{c1}$

$$\frac{\partial H^{c1}}{\partial u} = -\alpha u - \lambda\sqrt{x} = 0$$

$$\frac{\partial^2 H^{c1}}{\partial u^2} = -\alpha < 0 \quad \text{concave}$$

$$\boxed{u^*(t) = -\frac{\lambda(t)\sqrt{x(t)}}{\alpha}}$$

- $\dot{\lambda}(t) = -\dfrac{\partial H^{c1}}{\partial x} + r\lambda(t) = 1 + \dfrac{\lambda(t)u}{2\sqrt{x}} + r\lambda(t)$

CO-STATE SYSTEM PDE backward

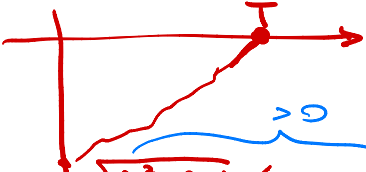
$$\begin{cases} \dot{\lambda}(t) = 1 - \dfrac{\lambda^2(t)\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}\,\alpha} + r\lambda(t) \\ \lambda(T) = 0 \end{cases}$$

Riccati eq

$$\left| \begin{array}{l} S(x(T)) = 0 \\ x(T) \in \mathbb{R} \end{array} \right. \Rightarrow \gamma = 0$$

$$\dot{\lambda}(T) = 1$$

$$\dot{\lambda} > 0$$



$$\exists!\ \lambda(t) = \frac{2\left(1 - e^{-\overbrace{\sqrt{x^2 + 2/\alpha}\,\underbrace{(T-t)}_{>0}}^{>0}}\right)^{>0}}{\underbrace{\underbrace{\left(x - \sqrt{x^2+2/\alpha}\right) e^{-c(T-t)}}_{<(x-\sqrt{\ })\,<\,(x+\sqrt{\ })} - \left(x + \sqrt{x^2+2/\alpha}\right)}_{<0}} \leq 0$$

?

$$\Phi(t) = u^* = -\frac{\lambda\sqrt{x}}{\alpha} \geq 0$$

best response

$$\boxed{\lambda(t) = 0 \iff t = T}$$

Player 1

$$\begin{cases} \dot{x}(t) = 1 + \varphi(t) + \dfrac{\lambda(t)\,x(t)}{\alpha} \\ x(0) = x_0 \end{cases}$$ Linear ODE

$\varphi(t) \Rightarrow$ strategy of P2 ?!.

$x(t) = \_\_\_\_$?

Such a ϕ is only a candidate

Apply ARROW's sufficiency Theorem

$$(H^{c1})^* = H^{c1}(x, u^*, \lambda, t) = v - x - \frac{\alpha}{2}\left(-\frac{\lambda\sqrt{x}}{\alpha}\right)^2 + \lambda\left(1 + v + \frac{\lambda\sqrt{x}}{\alpha}\sqrt{x}\right)$$

$$= v - x - \frac{\lambda^2 x}{2\alpha} + \lambda + \lambda v + \frac{\lambda^2 x}{\alpha}$$

linear in x $\Rightarrow$ concave in x

$\Rightarrow$ ARROW ✓

$\Rightarrow \phi$ is optimal strategy.

Player 2

$u(t) = \phi(t)$ player 1

$H^{c2}(x, v, \mu, t) = v - x + \mu\left(1 + v - \phi\sqrt{x}\right)$ linear in v

$= (1+\mu)v - x + \mu\left(1 - \phi\sqrt{x}\right)$

- $\operatorname{argmax}_{v \in [0,1]} H^{c2}$

$$\psi^* = v^* = \begin{cases} 0, & \mu < -1 \\ 1, & \mu > -1 \end{cases}$$

bang-bang sol

$\phi = -\dfrac{\lambda\sqrt{x}}{\alpha}$

- $\dot{\mu}(t) = -\dfrac{\partial H^{c2}}{\partial x} + r\mu(t) = 1 + \dfrac{\mu\phi}{2\sqrt{x}} + r\mu(t)$

$= 1 + r\mu(t) - \dfrac{\mu\lambda}{2\alpha}$

- $\mu(T) = 0$

$$\begin{cases} \dot{\lambda} = 1 + r\lambda - \dfrac{1}{2\alpha}\lambda^2 & \lambda(T) = 0 \\ \dot{\mu} = 1 + r\mu - \dfrac{1}{2\alpha}\mu\lambda & \mu(T) = 0 \end{cases}$$

$$\dot{\lambda} - \dot{\mu} = \frac{1}{2\alpha}\lambda(\mu - \lambda) + r(\lambda - \mu)$$

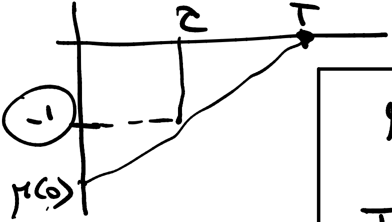
$$y(t) = \lambda - \mu$$

$$\begin{cases} \dot{y}(t) = \dfrac{\lambda y}{2\alpha} + r y = \underline{\left(\dfrac{\lambda}{2\alpha} + r\right)} y \Rightarrow y(t) = 0 \\ \underline{y(T) = 0} \end{cases}$$

$$\Downarrow$$

$$\boxed{\lambda(t) = \mu(t)}$$

Player 2



$$\boxed{\mu(0) < -1 \iff T > \frac{1}{c}\ln\left(\frac{2-\left(x-\sqrt{x^2+2/a}\right)}{2-\left(x+\sqrt{x^2+2/a}\right)}\right) = \tau}$$

$$\mu(0) > -1 \Rightarrow \tau = 0$$

$$u(t) = \psi(t) = \begin{cases} 0, & t \in [0,\tau) \\ 1, & t \in [\tau, T] \end{cases}$$

degenerate strategy not depending on the current state

Check optimality

$$-\frac{\lambda}{\alpha}\sqrt{x}$$

$$H^{*c2}(x, v^*, \mu, t) = \underset{0\ 1}{v^*} - x + \mu\left(x + \underset{0\ 1}{v^*} - \phi\sqrt{x}\right)$$

linear in x $\Rightarrow$ Arrow's Theorem holds

Open-Loop Nash EQ

(ϕ^*, ψ^*)

$$\underline{\phi^*(t)} = -\frac{\lambda(t)}{\alpha}\sqrt{x(t)}$$

$$\underline{\psi^*(t)} = \begin{cases} 0 & t \in [0, \tau) \\ 1 & t \in [\tau, T] \end{cases}$$

λ, τ see above

~~Player 2~~

$\underline{x^*(t)}$ state function

$$\begin{cases} \dot{x}(t) = 1 + v^*(t) - u(t)\sqrt{x(t)} \\ x(0) = x_0 \end{cases}$$

$$u^* = -\frac{\lambda\sqrt{x}}{\alpha}$$

In $[0, \tau)$ $\quad v^* = 0 = \psi^*$

$$\dot{x}(t) = 1 + \frac{\lambda x(t)}{\alpha}$$

$$x(t) = e^{\int_0^t \frac{\lambda(z)}{\alpha} dz} \left\{ x_0 + \int_0^t (1+\underset{0}{\psi}) e^{-\int_0^s \frac{\lambda(s)}{\alpha} ds} ds \right\}$$

$$\boxed{x(\tau)} \longrightarrow x_\tau$$

Player 2

In $[\tau, T]$ $\quad \sigma^2 = 1 = \varphi^2$

$$\begin{cases} \dot{x}(t) = 2 + \dfrac{\lambda(t)x(t)}{\alpha} \\ x(\tau) = x_\tau \end{cases}$$

$$x(t) = e^{\int_\tau^t \frac{\lambda(z)}{\alpha} dz} \left[x_\tau + \int_\tau^t 2 e^{-\int_\tau^s \frac{\lambda(z)}{\alpha} dz} ds \right]$$

//

MARKOVIAN NASH EQUILIBRIUM (MNE)

Alessandra Buratto

.

DIPARTIMENTO MATEMATICA

Università degli Studi di Padova

Nash equilibrium for a differential game

Finding a **Nash equilibrium** in a differential game with N players is equivalent to **solve** N **Optimal Control problems.**

- OPEN-LOOP NASH EQUILIBRIUM (OLNE)
  Pontryagin's Maximum Principle (1962) PMP

$$(\Phi^1(t), \Phi^2(t), \ldots, \Phi^i(t), \ldots, \Phi^N(t))$$

- MARKOVIAN NASH EQUILIBRIUM (MNE)
  Hamilton Jacobi Bellman ('50s) HJB

$$(\Phi^1(t, x(t)), \Phi^2(t, x(t)), \ldots, \Phi^i(t, x(t)), \ldots, \Phi^N(t, x(t)))$$

MARKOVIAN NASH EQUILIBRIUM (MNE)

Definition 4.1 The N-tuple $(\phi^1, \phi^2, \ldots, \phi^N)$ of functions $\phi^i : X \times [0, T) \mapsto \mathbb{R}^{m^i}$, $i \in \{1, 2, \ldots, N\}$, is called a Markovian Nash equilibrium if, for each $i \in \{1, 2, \ldots, N\}$, an optimal control path $u^i(\cdot)$ of the problem (4.1) exists and is given by the Markovian strategy $u^i(t) = \phi^i(x(t), t)$.

MARKOVIAN NASH EQ. (MNE)

Hamilton Jacobi Bellman Equation approach

Assume the existence and differentiability of Value function

$$V^i(x, t) = \max J^i$$

Hamilton Jacobi Bellman Equation approach

$$\text{maximize} \quad \int_{t_0}^{T} e^{-r(t-t_0)} F_0\left(x(t), u\left(t\right), t\right) dt + e^{-r(T-t_0)} S(x(T))$$

$$\begin{aligned} \text{subject to} \quad & \dot{x}\left(t\right) = f\left(x\left(t\right), u\left(t\right), t\right) \\ & x\left(t_0\right) = x \\ & u\left(t\right) \in \mathcal{U}(x(t), t) \end{aligned}$$

Looking for feedback strategies $\Phi(x(t), t)$

Value function $V\left(x, t\right)$

Hamilton Jacobi Bellman Equation approach: Value function

Definition

If $P_{x,t}$ has an optimal solution, then let $V(x,t)$ *be the optimal value*

$$V(x,t) = \max_{u(\cdot)} \left\{ \int_t^T e^{-r(s-t)} F_0(x(s;u), u(s), s)\, ds + e^{-r(T-t)} S(X(T;u)) \right.$$

$V(x,T) = m \int_T^T$

$= S(x(T))$

where $x(s;u)$ is the unique solution to

$$\begin{cases} \dot{x}(s) = f(x(s), u(s), s) \\ x(t) = x \end{cases}$$

associated with the control $u(\cdot)$

$$V(x,T) = S(x)$$

We assume the existence of the Value Function $V(x,t)$

Not differentiable Value Function

The differentiability of the Value Function is not assured, even if $F_0, f \in \mathcal{C}^\infty$

$$\text{maximize} \quad \int_t^T x(s)u(s)ds$$

$$\text{subject to} \quad \dot{x}(s) = u(s)$$
$$x(t) = x$$
$$u(s) \in [-1, 1]$$

$$J = \max \int_t^T x(s)\dot{x}(s)ds = \qquad = x(T)^2 - \underbrace{x(t)^2}_{x}$$

$$u^*(t) = \begin{cases} 1 \\ -1 \end{cases} \qquad \dot{x}(t) = \begin{cases} 1 \\ -1 \end{cases}$$

$$x^*(T) =$$

Not differentiable Value Function

$$x^*(t) = \begin{cases} x - (T-t) & x > 0 \\ x + (T-t) & x < 0 \end{cases}$$

$$V(x,t) = \frac{(T-t)^2}{2} + (T-t)|x|$$

NOT differentiable in $x = 0$ for all $t < T$.

In the HJB approach (suff. cond) differentiability is assumed.

Sufficiency conditions: Hamilton Jacobi Bellman Equation

Theorem (HJB equation)

Let $V : X \times [0, T] \mapsto \mathcal{R}$ be a continuously differentiable function which satisfies the HJB equation PDE

$\int F_0$ $\quad\lambda\quad$ $\dot{x}$

$$rV(x,t) - \frac{\partial V(x,t)}{\partial t} = \max_{u \in \mathcal{U}(x,t)} \left\{ F_0(x,u,t) + \frac{\partial V(x,t)}{\partial x} f(x,u,t) \right\} \tag{1}$$

$g(u)$ Hamiltonian

and the terminal condition

$$V(x,T) = S(x)$$

for all $(x,t) \in X \times [0,T]$. Let $\Phi(x,t)$ denote the set of controls $u \in \mathcal{U}(x,t)$ maximizing the RHS of 1. If $u(\cdot)$ is a feasible control path with corresponding state trajectory $x(\cdot)$ and if $u(t) \in \Phi(x(t),t)$ holds for almost all $t \in [0,T]$ then $u(\cdot)$ is an optimal control path. Moreover $V(x,t)$ is the optimal value (function) of problem $P_{x,t}$.

Hamilton Jacobi Bellman approach

a) Write HJB Eq assuming $V(x,t)$ differentiable

$$rV(x,t) - \frac{\partial V(x,t)}{\partial t} = \max_{u \in \mathcal{U}} \Big\{ \overbrace{F_0(x,u,t) + \frac{\partial V(x,t)}{\partial x} f(x,u,t)}^{g(u)} \Big\}$$

b) Find from RHS $\max_{u \in \mathcal{U}}\{\ \} \Rightarrow u^*(x,t,V_x)\ \forall t, \forall x$

c) Insert u^* in HJB eq. $\Rightarrow$ PDE

$$\begin{cases} rV(x,t) - V_t(x,t) = F_0(x,u^*,t) + V_x(x,t) f(x,u^*,t) \\ V(x,T) = S(x(T)) = S(x) \end{cases}$$

d) From c) try to guess a form for $V(x,t)$ (generally a polynomial with degree equal to the maximum degree between HJB and Boudary condition (see example))

n

Differential game Example: MARKOVIAN NASH EQ.

(Dockner p.87)

$$\begin{aligned} \max_{u\geq 0}\ J^1(u(\cdot)) &= \int_0^T e^{-rt}\left[v(t)-x(t)-\frac{\alpha}{2}u^2(t)\right]\,dt \\ \max_{v\in[0,1]}\ J^2(u(\cdot)) &= \int_0^T e^{-rt}\left[v(t)-x(t)\right]\,dt \\ \text{s.t.}\quad \dot{x}(t) &= 1+v(t)-u(t)\sqrt{x(t)}, \\ x(0) &= x^0 \end{aligned}$$

Assume $v(t)=\psi(x,t)$ for P2, find best response $\Phi(x,t)$ strategy for P1

Assume $u(t)=\Phi(x,t)$ for P1, find best response strategy $\psi(x,t)$ for P2

Player 2

$$rV - V_t = \max_{v\in[0,1]} \left\{ v - x + V_x \underbrace{(\alpha + v}_{h(v)} - \alpha\sqrt{x}) \right\}$$

ϕ

$$v = \begin{cases} 0 & 1 + V_x < 0 \\ 1 & 1 + V_x > 0 \end{cases}$$

$\vdots$

$\Rightarrow$ equivalent to OLNE

$\Rightarrow$ Degenerate markovian Nash equilibrium

Player 1: Look for $u \in \Phi(x,t)$, associated HJB equation

$$rV(x,t) - \frac{\partial V(x,t)}{\partial t} = \max_{u \in \mathcal{U}} \left\{ F_0(x,u,t) + \frac{\partial V(x,t)}{\partial x} f(x,u,t) \right\}$$

$$rV(x,t) - \frac{\partial V(x,t)}{\partial t} = \max_{u \geq 0} \left\{ \underbrace{v - x - \frac{\alpha}{2}u^2 + V_x(1 + v - u\sqrt{x})}_{g(u)} \right\}$$

b) $g'(u) = -\alpha u - V_x\sqrt{x} = 0$

$g''(u) = -\alpha < 0$

$$\boxed{u^* = \phi(x,t) = -\frac{V_x\sqrt{x}}{\alpha}}$$

(assume $V_x < 0$ $\Rightarrow u^*$ optimal)

c) Substitute

$$\begin{cases} rV - V_t = v - x - \dfrac{\alpha}{2}\left(\dfrac{V_x^2 x}{\alpha^2}\right) + V_x\left(1 + v + \dfrac{V_x^2 x}{\alpha}\right) \\ V(x,T) = 0 \end{cases}$$

$$\begin{cases} rV - V_t = \sigma - x + \dfrac{V_x^2\, x}{2\alpha} + V_x(1+\theta) \\ V(x,T) = 0 \end{cases}$$

$\deg_x 1$

$\deg_x 0$

degree $(V) = k$ $\quad \partial$ wrt $x \Rightarrow$ degree $V_x = k-1$

degree $(V_t) = k$

degree $(V_x^2) = 2(k-1)$

$k = 1 + 2(k-1) = 2k - 1 \Rightarrow k = 1$

$\Rightarrow$ Look for $V(x,t) = a(t)x + b(t) \Rightarrow$

$V_x = a(t)$

$V_t = a'(t)x + b'(t)$

HJB

$$\begin{cases} r\big(a(t)x + b(t)\big) - \big(a'(t)x + b'(t)\big) = \sigma - x + \dfrac{(a(t))^2 x}{2\alpha} + a(t)(1+\theta) \\ a(T) = 0,\ b(T) = 0 \end{cases}$$

...

$$\begin{cases} a'(t) = 1 + r\,a(t) - (a(t))^2/2\alpha \\ a(T) = 0 \end{cases}$$

ODEs

$a(t) \sim \lambda(t)$

$$\begin{cases} b'(t) = r\,b(t) - (1+\theta)\,a(t) \\ b(T) = 0 \end{cases}$$

Feedback Strategy

$$\Phi(x,t) = -\frac{V_x\sqrt{x}}{\alpha} = -\frac{a(t)\sqrt{x}}{\alpha}$$

The observed state of system at time t

Player 2,

$$\max \int_0^T e^{-rt}[v(t)-x(t)]\,dt$$

$$\begin{cases} \dot{x}(t) = 1 + v(t) + \lambda(t)x(t)/\alpha \\ x(0) = x_0 \\ v(t) \in [0,1] \end{cases}$$

$$H^{2c} = v - x + \underline{\underline{\tilde{\mu}}}\,(1 + v + \lambda x/\alpha)$$

$$\frac{\partial H}{\partial v} = 1 + \tilde{\mu} \qquad v(t) = \begin{cases} 0 & \tilde{\mu} < -1 \\ 1 & \tilde{\mu} > -1 \end{cases}$$

$$\dot{\tilde{\mu}} = -\frac{\partial H^{2c}}{\partial x} + r\tilde{\mu}(t) = 1 - \frac{\lambda}{\alpha}\tilde{\mu} + r\tilde{\mu} \quad \not\Rightarrow \quad \tilde{\mu} = \mu_{\text{OLN}}$$

Thesis MNE $\neq$ OLNE

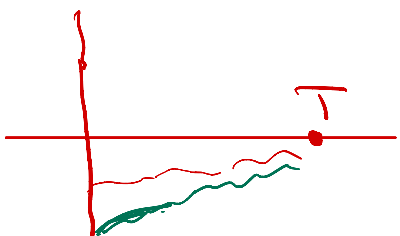
$$\tilde{\sigma} = \begin{cases} 0 & \tilde{\mu} < -1 \\ 1 & \tilde{\mu} > -1 \end{cases} \qquad \sigma = \begin{cases} 0 & \mu < -1 \\ 1 & \mu > -1 \end{cases}$$

To prove that MNE/OLNE we need to prove that

$$\tilde{\mu}(t) \neq \mu(t)$$

$$\dot{\tilde{\mu}} = 1 - \frac{\lambda}{\alpha}\tilde{\mu}(t) + x\tilde{\mu}(t) < 1 - \frac{\lambda}{2\alpha}\mu(t) + x\mu(t) = \dot{\mu}\Big|_{OLNE}$$

$\forall t \in [0, T]$ $\begin{cases} \dot{\tilde{\mu}}_{MNE}(t) < \dot{\mu}_{OLNE}(t) \\ \tilde{\mu}(T) = \mu(T) = 0 \end{cases}$



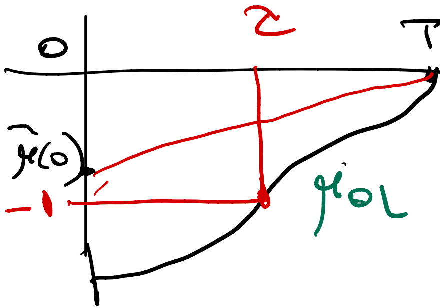
$\Rightarrow$ $\tilde{\mu}(t) > \mu(t) \neq 0$

MNE

$$(\Phi^N_{MNE}, \Psi^N_{MNE}) \neq (\Phi^N_{OL}, \Psi^N_{OL})$$

$$\tilde\mu(t) > \mu(t)_{OL}$$

$\sigma(t) = \psi(t) =$



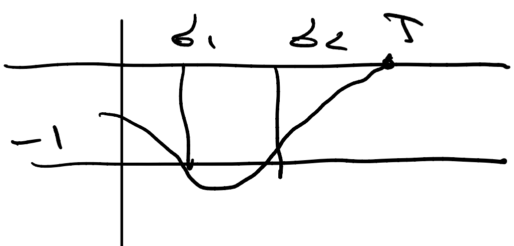
a) $\tilde\mu(0) > -1 \Rightarrow \forall t \quad \tilde\mu(t) > -1$

$$\Downarrow$$

$$\psi(t) = 1$$

b) $\tilde\mu(0) < -1$

$\exists!\ \tilde\tau$ s.t. $\tilde\mu(\tilde\tau) = -1 \qquad \tilde\tau < \tau$



c) $\tilde\mu(0) > 0 - 1$

$\exists$ 2 points $\psi = \begin{cases} 1 & t \in [0, \delta_1] \\ 0 & t \in (\delta_1, \delta_2) \\ 1 & t \in [\delta_2, T) \end{cases}$