LECTURE 2, March 2, 2023.

Example 3 : Hamilton-Jacobi equations.

(HJ) $\quad u_t + H(D_x u, x) = 0 \quad$ in $\Omega \times]0,T[$

$\Omega \subseteq \mathbb{R}^n$ open

evolutive & fully nonlinear.

- Analytical Mechanics $\quad H(p,x) = \frac{|p|^2}{2} + V(x)$ ↑potential

  H in convex in p.

- Optimal control : H-J-BELLMAN

  $$H(p,x) = \sup_a \{ -f(x,a)\cdot p - \ell(x,a) \}$$

  convex in p

- 0-sum differential games H-J-Isaacs

  $$H(p,x) = \sup_a \inf_b \{ -f(x,a,b)\cdot p - \ell(x,a,b) \}$$

  N.B. H not convex nor CONCAVE in p.

Ex. 4 STATIONARY H-J equations

$$H(Du, x) = 0 \quad \text{in } \Omega \qquad , u(x)$$

describe stationary sols. of (HJ)

Ex 4 bis EIKONAL EQ. in GEOMETRIC OPTICS

$$|Du| = n(x) \quad \text{refraction index}$$

$\left(|v| = \sqrt{\sum_{i=1}^n v_i^2} \right)$

- FERMAT principle

- Maxwell eqs. (~ book Born-Wolf).
- wave eq. ([E] = Evans ch. 4)

METHOD of CHARACTERISTICS

(1) $F(Du, u, x) = 0$ in $U \subseteq \mathbb{R}^n$

- Basic example & motivation: TRANSPORT EQ.

(TE) (CP)
$$\begin{cases} u_t + b(x) \cdot D_x u = 0 & \text{in } \mathbb{R}^n \times]0,T[\\ u(x,0) = g(x) & b, g \text{ data}, \end{cases}$$

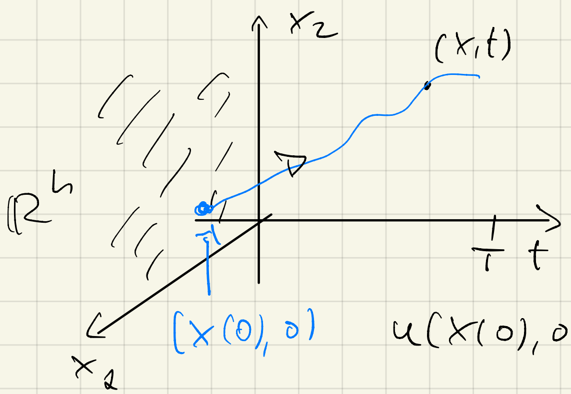
C^1 & $b \in Lip$ $\qquad b : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$.

(ODE) $\qquad \dot{x} = b(x)$.

Know that u solves (TE) $\iff$ $u(x(t),t) = \text{const}$ $\forall$ sols. $x(\cdot)$ of (ODE)

$\forall s \in [0,T]$.

$$u(x(t),t) = u(x(s),s) = g(x(0))$$



$$u(x(0),0) = g(x(0))$$

Def. $\Phi_t(x_0) := x(t; x_0) :=$ sol. of $\begin{cases} \dot{x} = b(x) \\ x(0) = x_0 \end{cases}$

FLOW of ODE.

FACT (known ....) $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, bijective, with C^1 inverse [$\Phi_t(\cdot)$ is a DIFFEOMORPHISM.].

Def $\quad u(x,t) := g(\Phi_t^{-1}(x)) \qquad (u(x,0) = g(x))$

- $u \in C^1$
- u is constant on traj of (ODE) by def.

$\Rightarrow$ u solves (TE)

$\Rightarrow$ I found a solut. of (TE), the unique sol.

Ex. $b(x) = b \in \mathbb{R}^n$ $\quad \Phi_t(x_0) = x_0 + bt \overset{?}{=} x$

$x_0 = x - bt \Rightarrow \Phi_t^{-1}(x) = x - bt \Rightarrow$ Sol. of (CP) is (b const.)

$u(x,t) = g(x - bt)$.

GENERAL CASE: Meth of charact. for

$$(1) \qquad (P) \begin{cases} F(Du, u, x) = 0 & \text{in } U. \\ u = g & \text{on } \Gamma \subseteq \partial U. \end{cases}$$

Supp. $\exists\, u \in C^2$ & look for curves in U

$\underline{X}(s) = (x_1(s), \dots, x_n(s))$ on which calculate

$u(\underline{X}(s)) = z(s)$, set $p(s) := Du(\underline{X}(s))$.

& look for a system of $2n+1$ (ODEs) for $(\underline{X}(\cdot), z(\cdot), p(\cdot))$

$$\dot{p}_i(s) = \frac{d}{ds} u_{x_i}(\underline{X}(s)) = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}(\underline{X}(s))\, \dot{x}_j(s) \qquad (\ast)$$

Supp. $F \in C^1$ & diff.te (1) w.r.t. x_i

$$(\times) \quad \sum_{j=1}^{N} F_{p_j}(Du, u, x) u_{x_j x_i} + F_z(Du, u, x) u_{x_i} + F_{x_i}(Du, u, x) = 0$$

Choose $\underline{X}(\cdot)$: $\dot{x}_j(s) = F_{p_j}(p(s), z(s), \underline{X}(s))$ $j = 1, \dots, N$

i.e. $\dot{\underline{X}}(s) = F_p(p(s), z(s), \underline{X}(s))$. Use $(\star) + (x)$

$$\dot{p}_i(s) = -F_z(p(s), z(s), \underline{X}(s))\, p_i - F_{x_i}(p(s), z(s), \underline{X}(s))$$

i.e. $\dot{p}(s) = -F_z(\ )\cdot p + F_x(\ )$

$\dot{z}(s) = Du(\underline{X}(s)) \cdot \dot{\underline{X}}(s) = p(s) \cdot F_p(p, z, \underline{X})$.

I found the system of characteristics :

$$\begin{cases} \dot{p} = -F_z(p, z, \underline{X})\, p - F_x(p, z, \underline{X}) & \text{(a)} \quad N \text{ eqs.} \\ \dot{z} = p \cdot F_p(p, z, \underline{X}) & \text{(b)} \quad 1 \text{ eq.} \\ \dot{\underline{X}} = F_p(p, z, \underline{X}) & \text{(c)} \quad N \text{ eqs.} \end{cases}$$

__N.B.__ These equations do not involve u !

__Def.__ Sols. of a+b+c are CHARACT. CURVES.

$\underline{X}(\cdot)$ projected characteristics. $(\subseteq U)$.

__Theorem__ (we proved) $F \in C^1$, $u \in C^2$ solves (1), if $\underline{X}(\cdot)$ solve (c)

with $p(s) = \nabla u(\underline{X}(s))$, & $z(s) = u(\underline{X}(s))$ $\Longrightarrow$

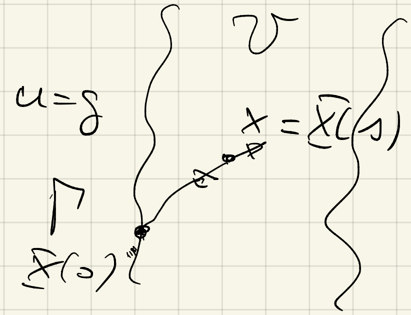
$p(\cdot)$ solves (a) & $z(\cdot)$ solve (b) $\forall s : \underline{X}(s) \in U$.

__Idea__ construct a sol. of (P) by solving (a-b-c)

+ "good" conditions on Γ :

$u(x) = z(X(s))$

with "suitable" initial conditions on (a-b-c)

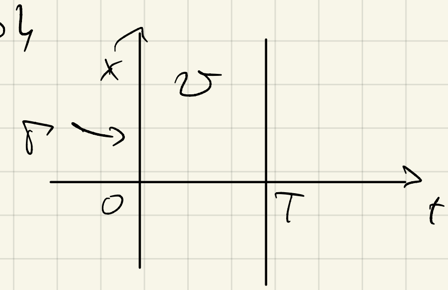


(p, z, X) solving (a, b, c)

For simplicity & brevity I restrict to $N = n+1$

$\mathcal{U} = \mathbb{R}^n \times]0,T[\quad (T \leq +\infty) \qquad \Gamma = \mathbb{R}^n \times \{0\}$

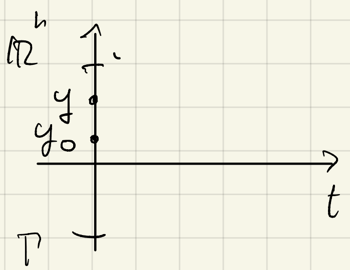
$X = (x,t), \quad p = (\underset{\in \mathbb{R}^n}{\bar{p}}, \underset{\in \mathbb{R}}{p_{n+1}})$



$$\text{(EV)}\ \text{(CP)} \begin{cases} u_t + G(D_x u, u, x, t) = 0 \\ u(x,0) = g(x) \end{cases}$$

i.e. w $F(p, z, X) = p_{n+1} + G(\bar{p}, z, x, t)$

Look for initial conditions for (a-b-c)



Fix y_0, $(y_0, 0) \in \Gamma$

For (c) $\begin{cases} x_i(0) = y_i & i = 1, \dots, n \\ x_{n+1}(0) = 0 \end{cases}$

For (b) $\quad z(0) = g(y)$

For (a) $\begin{cases} p_i(0) = g_{x_i}(y) & i = 1, \dots, n \\ p_{n+1}(0) = -G(Dg(y), g(y), y, 0) \end{cases}$

Consider the Cauchy for (a-b-c) + these initial conds.

If $G \in C^2$ r.h.s. of (a-b-c) is loc. Lip.

Then $\exists$ & $!$ $(p(y,s), z(y,s), X(y,s))$ sol of (a, b, c) + init. conds. in $\underset{y \in}{B(y_0, r_1)} \times]c, d[$, $\quad c < 0 < d$

Hypotheses $G \in C^3$, $g \in C^3$ $\Rightarrow$ vector fields in (abc) are C^2 & initial conds dep. in C^2 way on y $\Rightarrow$

by C^2 dependence of sols. of (ODE) from data

$$(y,s) \mapsto (P(y,s), Z(y,s), \bar{X}(y,s)) \in C^2(B(y_0, r_1) \times]c,d[).$$

Lemma $\bar{X}(\cdot,\cdot)$ is locally invertible, i.e., $\exists V \subseteq \mathbb{R}^{n+1}$ nbhd. of $(y_0, 0)$, $I = (a,b) \ni 0$, $W = B(y_0, r) \subseteq \mathbb{R}^n$ s.t. $\forall (x,t) \in V$ $\exists$ unique $s \in I$, $y \in W$! $(x,t) = \bar{X}(y,s)$.

Moreover the inverse is C^2.

Proof. Use INVERSE FN. THM. for $(y,s) \mapsto \bar{X}(y,s) \in C^2$.

Must check that Jac. $D\bar{X}(y_0, 0)$ has det $\neq 0$.

Then inverse $\exists$ locally & it is C^2.

$\bar{X}(y, 0) = (y, 0)$ the identity $y \mapsto \bar{X}(y,0)$ $\quad D_y \bar{X}(y_0, 0) = \begin{pmatrix} I_{n\times n} \\ 0 \dots 0 \end{pmatrix}$

$$\frac{\partial \bar{X}}{\partial s} = \dot{\bar{X}} \quad (c) \begin{cases} \dot{x}_i = G_{p_i}(P, Z, \bar{X}) \\ \dot{x}_{n+1} = 1 \qquad x_{n+1}(s) = s \\ \qquad\qquad x_{n+1}(0) = 0 \end{cases} \qquad F = p_{n+1} + G$$

$$D\bar{X}(y_0, 0) = \left(\begin{array}{c|c} I_{n\times n} & G_{p_i} \\ \hline 0 \dots 0 & 1 \end{array}\right) \qquad \det D\bar{X}(y_0, 0) = 1 \neq 0$$

$\boxed{\Xi}$

Def. $(\bar{Y}, S) =$ the local inverse of $\bar{X}(\cdot,\cdot)$ i.e.

$$(x,t) = \bar{X}(y,s) \iff y = \bar{Y}(x,t), \ s = S(x,t) = t$$

& def $\quad u(x,t) := z(\bar{Y}(x,t), t) \quad$ (D) $\quad (x,t) \in V$

N.B. 1. If (EV) = (TE) we recover what we did.

N.B. 2 $u \in C^2$ because Y & Z are C^2.

Thm The function u def. by (D) solves

$$(CP)\quad \begin{cases} u_t + G(D_x u, u, x, t) = 0 & \text{in } V \\ u(x,0) = g(x) & \text{in } V \cap \{t=0\}. \end{cases}$$

Rmk by construction.

Remains to prove that eq. holds.

- general case [EV, p.107-110]
- I will do $u_t + H(D_x u) = 0$

from P.L. Lions: "Gen. sol. of H-J eqs." Pitman 1982

Examples Ex. 1.

$$(LE)\qquad u_t + b \cdot D_x u + cu = e$$

$b \in \mathbb{R}^n$ const, c, e cont. fns of x & t.

$$G = b \cdot \bar{p} + c(x,t) z - e(x,t) \qquad F = p_{n+1} + G$$

$$(C)\quad \begin{cases} \dot{x}_i = b_i & x_i(0) = y_i & x_i(s) = y_i + s b_i \\ \dot{x}_{n+1} = 1 & x_{n+1}(0) = 0 & x_{n+1}(s) = s = t \end{cases}$$

$$\gamma(y,s) = (y + sb, s) \stackrel{?}{=} (x,t) \iff t = s \quad y = x - sb$$
$$=: Y(x,s)$$

Candidate sol. by (D) :

$$u(x,t) = z(\underline{Y}(x,t)) \qquad \text{Who is } z \text{ ?}$$

HW: Try to go on and find a formula for u ...