DIFFERENTIAL EQUATIONS 2023

Lect. 1 2/28/23

PLAN. Introduction to 1st. order NONLINEAR P.D.E. with applications.

Part 1. Hamilton-Jacobi equations & appls. to OPTIMAL CONTROL.

- CLASSICAL SOLUTIONS: METHOD OF CHARACTERIST. SINGULARITIES.
- LINK H-J eqs. $\leftrightarrow$ Analitycal mechanics. & CALCULUS of VARIATIONS... will suggest e notion of "variational solutions"
- WEAK SOLUTIONS: VISCOSITY SOLUTIONS THEORY.
- Introd. to OPTIMAL CONTROL via "DYNAMIC PROGRAMMING":

DYNAMICAL SYSTEM in CONTINUOUS TIME

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t)) & \text{in } \mathbb{R}^n \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}$$

α ← control.

WANT TO MINIMIZE $T > 0$

$$J = \int_0^T \ell(x(t), \alpha(t))\, dt$$

when $\alpha(\cdot)$ varies among ADMISSIBLE CONTROLS.

D.P. searches for FEEDBACK CONTROLS

⇝ a P.D.E. H-J-Bellman

REFS. : • L.C. EVANS. PDEs AMS 2^{nd} ed. 2010

• M.B. - I CAPUZZO DOLCETTA. Opt. control .....
Birkhäuser 2^{nd} ed 2008.

PART 2 TOPICS in DYNAMIC GAMES.
(DIFFERENTIAL)

• 2-person 0-sum GAMES '30 Von Neumann.

• N-person Games '50 NASH

• DIFFERENTIAL GAMES (2-person).

$$\dot{x}(t) = f(x(t), \alpha(t), \beta(t))$$

control of 2^{nd} agent.

control of 1^{st} agent

1st agent ⇝ MIN J_A cost.

2nd agent ⇝ MIN J_B "

• NON-ZERO SUM D. Games. LINEAR-QUADRATIC PROBLEMS.

⇝ SYSTEMS of N PDEs,
N = # of agents.

• 0-SUM DIFF GAMES. : $J_A = -J_B$

⟶ SINGLE PDE H-J-R.Isaacs (~60, ~90)

• MEAN-FIELD GAMES 2006-2007 J.M. Lasry
P.-L. Lions.

N agents, N large, $N \to \infty$ SYSTEM of 2 PDEs

$$\begin{cases} \text{HJB} \\ \text{div-form eq.} \end{cases}$$

REFS. 2nd part : • E.N. BARRON Game theory Wiley.

- A. Bressan ... Milan J. Math.
- [BCD] & Evans & Souganidis ... Indiana J.
- P. CARDALIAGUET. Lect. Notes of MFG.
- My notes.

PREREQUISITES : • ORDINARY DIFF. EQS.

- $\exists$ & ! of sols. of Cauchy probl.
- C^k dependence of solutions on data.

Refs. : write them at the end of talk.

- FUNCTIONAL ANALYSIS
  - weak convergence in $\mathbb{P}(A)$
  - Hahn-Banach $\longrightarrow$ convex functions.
  - Ascoli-Arzelà compactness for UNIF. CONVERG.

- OFFICE HOURS (RICEVIMENTO)
  - THURS. 10.30, or 14.30 Tell me.
  - e-mail for appointment ... zoom ...

- EXAMS. Can choose : • classical oral exam.

OR

- oral exam on a "monographic part".

OR

- focused on the exercises assigned as homeworks

GENERALITIES on PDEs.

ODE : $F\left(u^{(k)}, u^{(k-1)}, \dots, u'', u', u, x\right) = 0$ systems

given $\quad u = \text{unknown} \quad u(x) \quad x \in \mathbb{R}, \quad u: \mathbb{R} \supseteq I \to \mathbb{R}^d$

PDE : UNKNOWN scalar $u(x) = u(x_1, \dots, x_N)$, $x \in \mathbb{R}^n$

$u: \mathbb{R}^n \supseteq \overline{\mho} \to \mathbb{R} \quad \mho$ open. $\quad$ 2nd order :

given $F \quad F(D^2u, Du, u, x) = 0$

Hessian $\qquad$ grad $= \nabla u$

$F: \mathcal{S}_{N \times N} \times \mathbb{R}^N \times \mathbb{R} \times \overline{\mho} \to \mathbb{R}$

CLASSICAL 2^{nd} order LINEAR EQUATIONS :

Laplace / Poisson : $\displaystyle\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} =: \Delta u = f(x)$ ← datum.

Heat eq. : $\frac{\partial u}{\partial t} = u_t = \Delta_x u \qquad u(x, t)$

EVOLUTIVE.

Wave eq : $u_{tt} = \Delta_x u$

1st order EQS. : (1) $\quad F(Du, u, x) = 0$ in $\mho \subseteq \mathbb{R}^n$ open.

Given $F: \mathbb{R}^N \times \mathbb{R} \times \overline{\mho} \to \mathbb{R} \qquad$ UNKNOWN $u: \overline{\mho} \to \mathbb{R}$

GOAL : solve (1) with "BOUNDARY CONDITIONS" on u

GIVEN $\Gamma \subseteq \partial \mho$.

(BC) $u = g$ on Γ, $g : \Gamma \to \mathbb{R}$ GIVEN.

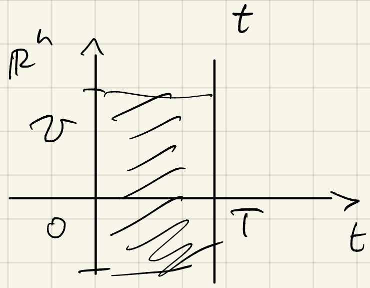
Ex 1. DIRICHLET PB.
$\Gamma = \partial \mathcal{V}$

$$\begin{cases} (1) & \text{in } \mathcal{V} \\ u(x) = g(x) & \forall x \in \partial \mathcal{V}. \end{cases}$$

Ex. 2 CAUCHY PB. for EVOLUTIVE EQS.:

$\mathbb{R}^N \ni x = (x_1, \dots, x_n, t)$ $\qquad N = n+1$

$$u_t + H(D_x u, u, x, t) = 0 \quad \text{in a strip } \mathcal{V}$$



e.g.: $\mathcal{V} = \mathbb{R}^n \times]0, T[$ $\qquad T \leq +\infty$

$\Gamma = \mathbb{R}^n \times \{0\} = \text{"}\{t = 0\}\text{"} \subsetneq \partial \mathcal{V}$

$u(x, 0) = g(x)$ INITIAL CONDITION.

CLASSIFICATION of I^{st} order PDEs.

- LINEAR EQS.: $\displaystyle\sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + b(x) u = f(x)$ ↑ forcing term

$\underbrace{a_i, b}$, f data coefficients.

- SEMILINEAR EQS.: $\displaystyle\sum_i a_i \frac{\partial u}{\partial x_i} = f(x, u)$

- QUASILIN. EQS.: $\displaystyle\sum_i a_i(x, u) \frac{\partial u}{\partial x_i} = f(x, u)$

- FULLY NONLINEAR is (1) $F(D_x u, u, x) = 0$ with F NOT reducible to the previous cases.

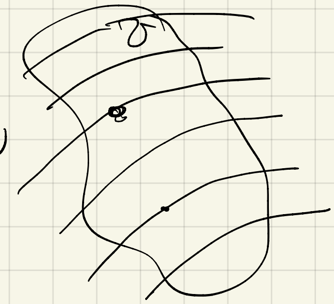
Examples & motivations.

1 Transport eq. : Particles moving with dynamics

(ODE) $\dot{x}(t) = b(x(t))$ $\dot{x} = x'$

given $b : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz $\left(|b(x) - b(y)| \le L|x-y| \right)$

Supp. $u \in C^1(\mathcal{V})$, $\mathcal{V} \subseteq \mathbb{R}^{n+1}$ CONSTANT on TRAJECTORIES of (ODE). : $u(x(t),t) = \text{const}$



$$0 = \frac{d}{dt} u(x(t),t) = D_x u(x(t),t) \cdot \underset{\overset{\shortparallel}{b(x(t))}}{\dot{x}(t)} + u_t(x(t),t)$$

If trajectories of (ODE) $(x(t),t)$ fill $\mathcal{V}$ $\Rightarrow$

u sats. TRANSPORT EQ.

$$\text{(TE)} \qquad \frac{\partial u}{\partial t} + D_x u \cdot b(x) = 0 \quad \text{in } \mathcal{V}$$

EVOLUTIVE, LINEAR.

Viceversa : if $u \in C^1(\mathcal{V})$ sat. (TE) $\Rightarrow$

$u(x(t),t) = \text{const}$ $\forall$ traject. ☒

Ex.2 : SCALAR CONSERVATION LAW.

$$\text{(CL)} \qquad \frac{\partial u}{\partial t} + \text{div}_x f(u) = 0 \quad , \quad f : \mathbb{R} \to \mathbb{R}^n \in C^1$$

EVOLUT.

"in div form" $\qquad \overset{\shortparallel}{\displaystyle\sum_{i=1}^{n} \frac{df_i}{du}(u)\, u_{x_i}}$ QUASILIN.

$n = 1$ $\qquad u_t + f(u)_x = 0 \qquad u_t + f'(u)\, u_x = 0$

Ex 2 bis $\qquad f(u) = u^2 \qquad u_t + u u_x = \varepsilon u_{xx} \quad \varepsilon \to 0$

INVISCID BURGER EQ. in GAS DYNAMICS.

$u(x,t)$ = density of mass of the gas (electrical charge, cars in a street ....)

DERIVATION $\quad \Omega \subseteq \mathbb{R}^n$ open set.

density of flux

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = -\int_{\partial\Omega} f(u)\cdot n_e \, d\sigma$$

$$\parallel$$

$$\int_\Omega u_t(x,t)\,dx = -\int_\Omega \operatorname{div}_x f(u)\,dx \qquad \forall \Omega \subseteq \mathbb{R}^n$$

NOTE: HERE $f = f(u)$ NOT $f = f(Du)$

$$\Rightarrow \frac{\partial u}{\partial t} = -\operatorname{div}_x f(u) \qquad (CL)$$

in $U = \operatorname{dom} u$

$$\int_\Omega \underbrace{(u_t + \operatorname{div}_x f(u))}_{h(x,t)}\,dx = 0$$

if $h(\bar{x},\bar{t}) > 0$

$\exists B_r$: $h > 0$ in B_r.

References for C^k dependence of the solution of (ODE) with respect to initial data:

- $k = 1$: A. Bressan - B. Piccoli: Introduction to the mathematical theory of control AIMS 2007
- $k \geq 1$: V. Arnold: Ordinary differential equations, MIT Press 1978 (also in Italian & French)
- $k = 1$, without proof: C. Pagani, S. Salsa: Analisi Mat. 2 2nd ed., Zanichelli 2016.