February 23rd, 2023 23 02 '23

Static games in normal form
Choice of strategies

- Dominating strategies
- MiniMax Theorem (von Newmann)

but Saddle points existence not guaranteed

Nash Equilibrium

A set of strategies constitutes a Nash equilibrium if no single player is interested in changing his strategy unless one of the other players changes his own.

$$(\varphi_1^N, \varphi_2^N, \ldots, \varphi_m^N) \rightarrow \text{Nash}$$

players

That is:

Keeping the choices of other players fixed,
Nobody is interested in changing his own.

max J

$$J_i(\varphi_1^N, \ldots \varphi_{i-1}^N, \varphi_i^N, \varphi_{i+1}^N, \ldots, \varphi_m^N) \geq J_i(\varphi_1^N, \ldots \varphi_{i-1}^N, \varphi_i, \varphi_{i+1}^N, \ldots, \varphi_m^N)$$

$\forall i$

$\forall \varphi_i \in \mathcal{U}_i$

feasible strategy/action

Example with No saddle point but there exists 1 Nash equilibria

Set of strategies (a, **β**) s.t.:
Knowing that G1 playes **a** then for G2 has not choise (convenience) but to play **β**
Knowing that G2 playes **β** then for G1 has not choise (convenience) but to play **a**

$$
\begin{array}{ccc}
 & & \text{G2} \\
 & \boldsymbol{\alpha} & \boldsymbol{\beta} \\
\text{G1} \quad a & (5, 5) & (3, 3) \\
b & (2, 2) & (0, 0)
\end{array}
$$

Example with No saddle point but there exist 2 Nash equilibria

Set of strategies (a, **β**) s.t.:
Knowing that G1 playes **a** then for G2 has not choise (convenience) but to play **β**
Knowing that G2 playes **β** then for G1 has not choise (convenience) but to play **a**

G2

$$
\begin{array}{ccc}
 & \boldsymbol{\alpha} & \boldsymbol{\beta} \\
\text{G1} \quad a & (-3, -2) & (2, 0) \\
b & (0, 2) & (1, 1)
\end{array}
$$

CAR RACE / Chicken game

1 \ 2	Deviate	Straight	
D	(5, 5)	$\boxed{(1, 8)}$	1
S	$\boxed{(8, 1)}$	(-2, -2)	-2
	1	-2	

$\exists$ Dom Strat

$\nexists$ Saddle point

2 Nash eq.

MATCHING PENNY ~ Scissor Paper Stone

A \ M	H	T	
H	(1, -1)	(-1, 1)	-1
T	(-1, 1)	(1, -1)	-1

$\nexists$ Dom Strat

$\nexists$ Saddle point

$\nexists$ Nash eq. for PURE STRATEGIES

$\exists$ mixed strategies

$$A \begin{cases} H & T \\ p & (1-p) \end{cases} \qquad M \begin{cases} H & T \\ q & (1-q) \end{cases}$$

Expected

$$\underline{\underline{A}} \begin{cases} 1 & -1 & -1 & 1 \\ pq & p(1-q) & (1-p)q & (1-p)(1-q) \end{cases}$$

$E(\pi_A) = \ldots$

$$\frac{\partial E(\pi_A)}{\partial p} \rightarrow q = \frac{1}{2},\ p = \frac{1}{2}$$

Nash equilibria for static games

Existence of Nash equilibrium Kakutani fixed point theorem for multivalued maps. Consequence of the classical Brouwer fixed point theorem.

In a zero-sum game, if a Nash equilibrium exists, then all Nash equilibria yield the same payoff V (value of the game) (von Neumann)

Nash Equilibrium Existence Theorem (1950)

In a finite game there exists **at least one** Nash equilibrium (eventually **mixed strategies**)

finite number
- players
- actions

$$
\begin{array}{ccc}
 & & \text{P2} \\
 & & \begin{array}{cc} y1 & y2 \end{array} \\
\text{P1} & \begin{array}{c} z1 \\ z2 \end{array} & \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}
\end{array}
$$

(u_i^N , u_{-i}^N) Nash equilibrium

$$J^i (u_i^N , u_{-i}^N) > J^i(u_i , u_{-i}^N) \text{ for all } u_i \in U^i$$

Nash Equilibrium

- There might be other combination of strategies that increase the payoff of some players without reducing the payoffs of the others. Or, more, that increase the payoff of all players: **Prisoners' dilemma**.

Prisoners' Dilemma

- If **only one** confesses, **and puts the blame on the other one**, then he is set free and the other will be sentenced to 6 years of jail;
- If **both** confess, they will be sentenced to 5 years.
- If **neither one** confesses, they will be sentenced to 1 year.

$$\begin{array}{ccc} & \begin{array}{cc} C \quad\quad & \quad\quad NC \end{array} & \text{Min A} \\ \begin{array}{c} C \\ NC \end{array} & \begin{bmatrix} (-5\ ,\ -5) & (0\ ,\ -6) \\ (-6,\ 0) & (-1\ ,\ -1) \end{bmatrix} & \begin{array}{l} -5 \longleftarrow \\ -6 \end{array} \end{array}$$

Prisoners' Dilemma A

	C	*NC*	Min A
C	(-5 , -5)	(0 , -6)	-5
NC	(-6, 0)	(-1 , -1)	-6

Prisoners' Dilemma B

$$
\begin{array}{ccc}
 & \mathit{C} & \mathit{NC} \\
\begin{array}{c} \mathit{C} \\ \mathit{NC} \end{array} & \multicolumn{2}{c}{\begin{bmatrix} (-5, -5) & (0, -6) \\ (-6, 0) & (-1, -1) \end{bmatrix}} \\
\mathit{Min\ B} & -5 & -6
\end{array}
$$

Prisoners' Dilemma

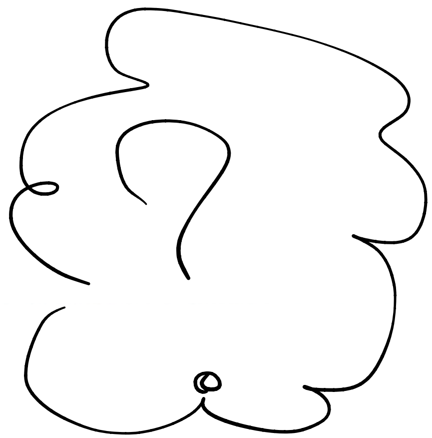
$$
\begin{array}{c c}
 & \begin{array}{cc} \quad A \quad & \quad NA \quad \end{array} \\
\begin{array}{c} A \\ NA \end{array} & \begin{bmatrix} (-5\ ,\ -5) & (0\ ,\ -6) \\ (-6,\ 0) & \boxed{(-1\ ,\ -1)} \end{bmatrix}
\end{array}
$$

MaxMin A

← Cooperative solution

Max Min of B

∃! NASH eq (-5, -5)



Prisoners' Dilemma
Nash equilibrium

$$\begin{array}{cc} & \begin{array}{cc} A & NA \end{array} \\ \begin{array}{c} A \\ NA \end{array} & \begin{bmatrix} (-5\ ,\ -5) & (0\ ,\ -6) \\ (-6,\ 0) & \boxed{(-1\ ,\ -1)} \end{bmatrix} \end{array}$$

Nash Equilibrium

- Existence and uniqueness is not guaranteed
  → There might exist more that one NE

- It gives solution when there might be uncertainty
- Each player does what is better for him (noncooperative)
- It might not be the better solution for everybody.
- Someone might increase his payoff moving far from the equilibrium.

Nash Equilibrium might not be Pareto Optimum.

– Giancarlo Gambarelli Bergamo

Nash equilibrium

Noncooperative simultaneous game

- Symmetric Information structure

$$\left.\begin{array}{lcl} \underset{u_1 \in U^1}{\textit{Max}}\ J_1(u_1,u_2) & \Rightarrow & u_1^{BR}=u_1(u_2) \\[2em] \underset{u_2 \in U^2}{\textit{Max}}\ J_2(u_1,u_2) & \Rightarrow & u_2^{BR}=u_2(u_1) \end{array}\right\} \quad (u_1^N,u_2^N)$$

Stackelberg game

Noncooperative sequential game

- Asymmetric information structure

1. LEADER: declares his action u_L
2. FOLLOWER: computes his best response $u_F(u_L)$ (to any Leader's strategy u_L)
3. LEADER: computes his optimal Stackelberg strategy u_L^S
4. FOLLOWER: adjust his strategy to obtain the Stackelberg strategy u_F^S

$$\mathop{Max}_{u_F \in U^F} J_F(u_L, u_F) \Rightarrow u_F^{BR} = u_F(u_L) \Rightarrow \boxed{\mathop{Max}_{u_L \in U^L} J_L(u_L, u_F(u_L))}$$

$$(u_L^S, u_F^S)$$

Coordination game

Cooperative simultaneous game

A Franc.
7 10
J_1 J_2
u_1^2 u_2^2
(u_1^c, u_2^c)

- Symmetric information structure

$$\max_{u_1,u_2 \in U^1 \times U^2} J_1(u_1,u_2)+J_2(u_1,u_2)$$

$(u_1^c, u_2^c) \longrightarrow J^c$ 18 ? How can we share it ?

Bargaining — Rationality / fair

Nash Bargaining Solution $\frac{18-(7+10)}{m} = 1/2$

A : 7+1/2 F : 10+1/2

Example Cournot duopoly
static game with infinite strategy sets

price

production

$J_1=(\alpha-\beta(Q_1+Q_2))Q_1-K_1Q_1^2$ profits $J_2=(\alpha-\beta(Q_1+Q_2))Q_2-K_2Q_2^2$

NASH: $(Q_1^N,Q_2^N)=\left(\dfrac{\alpha}{2K_1+3\beta},\dfrac{\alpha}{2K_2+3\beta}\right)$

Symm.case $\alpha=\beta=1, K_i=0 \Rightarrow (Q_1^N,Q_2^N)=(1/3,1/3)$. $J_1^N=J_2^N=1/9$

STACKELBERG: $(QLS,QFS)=\left(\dfrac{\alpha\left(1-\dfrac{\beta}{2(K_F+\beta)}\right)}{2(K_L+\beta)-\beta^2/(K_F+\beta)},\dfrac{\alpha-\beta Q_L{}^S}{2(K_F+\beta)}\right)$

Symm. case $\alpha=\beta=1, K_i=0 \Rightarrow (Q_1^S,Q_2^S)=(1/2,1/4)$ $J_L^S=1/8$, $J_F^S=1/16$

$\frac{1}{16} < \frac{1}{9} < \frac{1}{8}$

$J_F < J_N < J_L$

COOPERATIVE: Symm. case $\alpha=\beta=1, K_i=0 \Rightarrow$ $J^C=2/9=J_1^N+J_2^N$

IN GENERAL $(\alpha,\beta\in\mathbb{R}, K_i=0) \Rightarrow$ $J^C>J_1^N+J_2^N$

$\boxed{J^C > J_1^N+J_2^N}$

Nash

$\max_{Q_1} J_1(Q_1, Q_2)$ quadratic, concave

$$\left\{\begin{array}{l} \dfrac{\partial J_1}{\partial Q_1} = \alpha - \beta(Q_1+Q_2) - \beta Q_1 - 2K_1 Q_1 = 0 \\ \dfrac{\partial J_2}{\partial Q_2} = \alpha - \beta(Q_1+Q_2) - \beta Q_2 - 2K_2 Q_2 = 0 \end{array}\right.$$

$$-\beta Q_1 - 2K_1 Q_1 + \beta Q_2 + 2K_2 Q_2 = 0$$

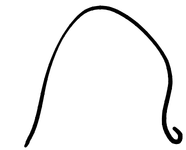
$$(\beta + 2K_2) Q_2 = (\beta + 2K_1) Q_1$$

Stackelberg $\dfrac{\partial J_F}{\partial Q_F} = \alpha - \beta(Q_L + Q_F) - \beta Q_F - 2K_F Q_F = 0$

$$Q_F^{BR}(Q_L) = \frac{\alpha - \beta Q_L}{2\beta + 2K_F}$$

$\rightarrow Q_F^{BR}$ plug in J_L

$$J_L = \left(\alpha - \beta\left(Q_L + \frac{\alpha - \beta Q_L}{2\beta + 2K_F}\right)\right)Q_L - K_L Q_L^2$$



max $\quad \dfrac{\partial J_L}{\partial Q_L} = 0$

Q_L^S