Open-Loop Nash Equilibrium

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OPEN-LOOP NASH EQUILIBRIUM (OLNE)

Definition 4.2 The N-tuple $(\phi^1, \phi^2, \ldots, \phi^N)$ of functions $\phi^i : [0, T) \mapsto \mathbb{R}^{m^i}$, $i \in \{1, 2, \ldots, N\}$, is called an open-loop Nash equilibrium if, for each $i \in \{1, 2, \ldots, N\}$, an optimal control path $u^i(\cdot)$ of the problem (4.1) exists and is given by the open-loop strategy $u^i(t) = \phi^i(t)$.

OPEN-LOOP NASH EQ. (OLNE)

Pontryagin Maximum Principle (1962)

Hamiltonian function

$$H^I(x, u^I, \lambda, t) = e^{-r^i t}\, F^i(\ ) + \lambda \cdot f^i(\ )$$

$\lambda(t) : [0, t] \mapsto \mathcal{R}^n$

Current Value Hamiltonian

$$H^{iC}(x, u^I, \lambda, t) = \ F^i(\ ) + \lambda \cdot f^i(\ )$$

Differential game Example: OPEN-LOOP NASH EQ.

(Dockner p.87)

$$\max_{u \geq 0} \; J^1(u(\cdot)) = \int_0^T e^{-rt} \left[v(t) - x(t) - \frac{\alpha}{2} u^2(t) \right] dt$$

$$\max_{v \in [0,1]} \; J^2(u(\cdot)) = \int_0^T e^{-rt} \left[v(t) - x(t) \right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = 1 + v(t) - u(t)\sqrt{x(t)},$$

$$x(0) = x^0$$

Assume $v(t) = \psi(t)$ for P2, find best response strategy for P1
$H^{C1}(x, u, \lambda, t) =$

Assume $u(t) = \Phi(t)$ for P1, find best response strategy for P2
$H^{C2}(x, u, \lambda, t) =$

Optimal control problem discount factor (DOC)

$$\begin{aligned} \text{maximize} \quad J(\underline{u}) &= \int_0^T e^{-\rho t} F_0(\underline{x}(t), \underline{u}(t), t)\, dt + e^{-\rho T} S(\underline{x}(T)) \\ \text{subject to} \quad \dot{\underline{x}}(t) &= \underline{f}(\underline{x}(t), \underline{u}(t), t) \\ \underline{x}(0) &= \underline{x}^0 \\ x_i(T) &= x_i^1 \quad i = 1, \dots, l \\ x_i(T) &\geq x_i^1 \quad i = l+1, \dots, m \\ x_i(T) &\in \Re \quad i = m+1, \dots, n \\ \underline{u}(t) &\in \Omega \end{aligned}$$

associated **current value** Hamiltonian function

$$H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \underline{q} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

$$H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \sum_{i=1}^{i=n} q_i f_i(\underline{x}(t), \underline{u}(t), t)$$

Pontryagin Maximum principle (discount factor)

Theorem

Let $u^*(t)$ be a piecewise continuous control defined on $[0, T]$ which solves problem (DOC) and let $x^*(t)$ be the associated optimal path. Then $\exists$ $n+1$ constants $p_0, \gamma_1, \ldots, \gamma_n \in \Re$ and a continuous and piecewise continuously differentiable function $q(t) = (q_1(t), \ldots, q_n(t))$ such that $\forall t \in [0, T]$

- $(p_0, \gamma_1, \ldots, \gamma_n) \neq (0, 0, \ldots, 0)$
- *$u^*(t)$ maximizes $H^C(x^*(t), u, q(t), t)$ for all $u \in \Omega$*
- *Excepts at the points of discontinuities of $u^*(t)$*

$$\dot{q}_i(t) = -\frac{\partial H^C(x^*(t), u^*(t), q(t), t)}{\partial x_i} + \rho\, q(t)$$

- $p_0 \in \{0, 1\}$ *for $i = 1, \ldots, n$*
- *Transversality conditions* *($\rightarrow$ next page)*

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Theorem

- *Transversality conditions*

$$q_i(T) = p_0 \frac{\partial S(x^*(T))}{\partial x_i} + \gamma_i, \quad i = 1, \dots n$$

where

$$\begin{array}{lll} \gamma_i \in \Re, & i = 1, \dots, l & \text{if } x_i^*(T) = x_i' \\ \gamma_i \geq 0 & i = l+1, \dots, m & \text{if } x_i^*(T) \geq x_i' \\ & \gamma_i(x_i^*(T) - x_i') = 0 & \\ \gamma_i = 0 & i = m+1, \dots, n & \text{if } x_i^*(T) \in \Re \end{array}$$

Player 1

Player 2