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Linear Quadratic and Linear State games

Alessandra Buratto

.

DIPARTIMENTO MATEMATICA

Università degli Studi di Padova

Games with special structures

Desirable characteristics:

- Analytical tractability
- Time consistency

Special structures:

- Linear-Quadratic (LQ)
- Linear-State (LS)
- Exponential (E) – can be transformed into (LS)

Linear dynamics and quadratic payoff functions

Ex: 2 players

$$\min J^1 \quad = \frac{1}{2}\int_0^T e^{-rt}\left[g_1(x(t))^2 + g_2(u_1(t))^2\right] dt + \alpha_1(x(T))^2$$

$$\min J^2 \quad = \frac{1}{2}\int_0^T e^{-rt}\left[m_1(x(t))^2 + m_2(u_2(t))^2\right] dt + \alpha_2(x(T))^2$$

$$\dot{x}(t) = a(t)x(t) + b(t)u_1(t) + c(t)u_2(t)$$

$$x(0) = 0$$

$$u_1(t), u_2(t) \in \mathcal{R}$$

Observation: Here there are homogeneous cost functions just to simplify computation

Equilibria for LQ games

- **Analytical tractability**
  OLNE and MNE easy to be obtained analytically

- **Time consistency**
  OLNE NOT subgame perfect
  MNE subgame perfect

Bressan (2011)
Dockner (2000)
Engwerda (2005) - LQ Dynamic Optimization and Differential Games

OLNE for LQ game: Example (constant coefficients)

$$
\begin{aligned}
H_1^C(x, u_1, p_1, t) &= -\tfrac{1}{2}(g_1 x^2 + g_2 u_1^2) + p_1(ax + bu_1 + cu_2) \\
H_2^C(x, u_1, p_1, t) &= -\tfrac{1}{2}(m_1 x^2 + m_2 u_2^2) + p_1(ax + bu_1 + cu_2)
\end{aligned}
$$

If $T < +\infty \quad p_1(T) = 0, \quad p_2(T) = 0$

$\max_{u_i \in \mathcal{R}} H_i^C$

$$
u_1(t) = \frac{b}{g_2} p_1(t) \qquad u_2(t) = \frac{c}{m_2} p_2(t)
$$

$$
\begin{aligned}
\dot{p}_1(t) &= -\frac{\partial H_1^C}{\partial x} = g_1 x(t) + (r - a) p_1(t) \\
\dot{p}_2(t) &= -\frac{\partial H_2^C}{\partial x} = m_1 x(t) + (r - a) p_2(t)
\end{aligned}
$$

co-state eq's are coupled with the dynamics

Canonical System

$$
\begin{cases}
\dot{x}(t) = ax(t) + \frac{b^2}{g_2}p_1(t) + \frac{c^2}{m_2}p_2(t), & x(0) = 0 \\
\dot{p}_1(t) = g_1 x(t) + (r-a)p_1(t), & p_1(T) = 0 \\
\dot{p}_2(t) = m_1 x(t) + (r-a)p_2(t), & p_2(T) = 0
\end{cases}
$$

$$
\underbrace{\begin{pmatrix} \dot{x} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix}}_{\dot{Y}} = \underbrace{\begin{pmatrix} a & b^2/g_2 & c^2/m_2 \\ g_1 & r-a & 0 \\ m_1 & 0 & r-a \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ p_1 \\ p_2 \end{pmatrix}}_{Y}
$$

can be solved analitically look for eigenvalues

$\det(A - \lambda I) = (\lambda - r + a)^2(\lambda - a) - (\lambda - r + a)M = 0$

where $M = c^2(m_1/m_2) + b^2(g_1/g_2) > 0$

eigenvalues

$$Y = \sum_{i=1}^{3} v_i e^{\lambda_i t}$$

The corresponding eigenvectors

$$\lambda_1 = \frac{r}{2} - \sqrt{\frac{r^2}{4} - a(r-a) + M}$$
$$\lambda_2 = \frac{r}{2} + \sqrt{\frac{r^2}{4} - a(r-a) + M} > 0$$
$$\lambda_3 = r - a$$

$$x(t) = h(g_1, g_2, m_1, m_2, x_0)$$

OLNE

$$(\Phi_1, \Phi_2) = \left(\frac{b}{g_2} \frac{w_{21}}{w_{11}} e^{\lambda_1 t} x_0, \frac{c}{m_2} \frac{w_{21}}{w_{31}} e^{\lambda_1 t} x_0 \right)$$

it is NOT markovian , NOT subgame perfect

See paper Li Yu et al "A new feedback form of open-loop stackelberg strategy in a general linear-quadratic differential game." (2022)

MNE for LQ games

Hamilton Jacoby Bellman equation turns out to be quadratic in x

Quadratic value function:

Riccati eq.

Homogeneous case:

$V_i(x,t) = \frac{1}{2} v_i(t) x^2$ if $T < +\infty$ finite time horizon ODEs

$V_i(x,t) = \frac{1}{2} \underbrace{v_i}_{\xi} x^2$ if $T = +\infty$ infinite time horizon (steady stat solutions) algebraic eq.

Non Homogeneous case: $V(x,t) = \underbrace{x^2}_{\alpha(t)} + \underbrace{\beta(t)}_{\alpha(t)} x + \underbrace{\gamma(t)}_{\alpha(t)}$

Linear feedback strategies

$$\boxed{(\Phi_1(x,t), \Phi_2(x,t)) = \left(\frac{b}{g_2} v_1(t)\; x, \frac{c}{m_2} v_2(t)\; x \right)}$$

Definition (Linear - State games)

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$$J^i = \int_0^T e^{-rt} L_i(x(t), \underset{i}{u_1}(t), \underset{-i}{u_2}(t), t)\ dt + e^{-rT} S_i(x(T))$$

$$\dot{x}(t) = f(x(t), u_1(t), u_2(t), t)$$

Define $\tilde{H}_i(x, u_1, u_2, p_i, t) = L_i(x(t), u_1, u_2, t) + p_i f(x(t), u_1, u_2, t)$

i)

$$\frac{\partial^2 \tilde{H}_i}{\partial x^2}(x, u_1, u_2, p_i, t) = 0, \qquad \frac{\partial^2 S_i(x)}{\partial x^2} = 0$$

linear w.r.t. x

ii)

$\hat{u}_i$: candidate

whenever

$$\frac{\partial \tilde{H}_i}{\partial u_i}(x, u_1, u_2, p_i, t) = 0 \Rightarrow \frac{\partial^2 \tilde{H}_i}{\partial u_i \partial x}(x, u_1, u_2, p_i, t) = 0$$

Linear state game: Sufficient conditions

Proposition (Sufficient conditions)

If there is no multiplicative interaction between the state $x(t)$ and the controls $u_i(t)$, then the game is Linear-State

$$\frac{\partial^2 \tilde{H}_i}{\partial u_1 \partial x}(x, u_1, u_2, p_i, t) = \frac{\partial \tilde{H}_i^2}{\partial u_2 \partial x}(x, u_1, u_2, p_i, t) = 0 \Rightarrow \textit{ Linear State}$$

Example: additive contribution

$L_i(x(t), u_1, u_2, t) = c_i(t)x(t) + k(u_1(t), u_2(t), t)$

$\dot{x}(t) = A(t)x(t) + g_1(u_1(t), u_2(t), t) = f(x(t), u_1, u_2, t)$ no $x \cdot u$

$S_i(x) = W_i x$ Linear in x

Linear state game: Sufficient conditions example

$x(t)$ stock of knowledge
$u_i(t)$ investment of player i in public knowledge

$$\dot{x}(t) = u_1(t) + u_2(t) - ax(t)$$

$$\max \ J_i = \int_0^T e^{-rt}\left[x(t) - k_i(u_i(t))\right] dt + e^{-rT} W_i x(T)$$

$$\tilde{H}_i(x, u_1, u_2, p_i, t) = x - k_i(u_i) + p_i(u_i + u_j - ax)$$

$$\frac{\partial \tilde{H}_i}{\partial u_i} = -k_i'(u_i) + p_i \quad \overset{\text{suff.}}{\Rightarrow} \quad \boxed{k_i'(u_i) = p_i(t)}$$

assuming $-k_i'' < 0$ $\quad k_i'' > 0$ $\quad \tilde{H}$ $\frown$

marginal cost of investment= marginal utility

$$\dot{p}_i(t) = -\frac{\partial \tilde{H}_i}{\partial x} + rp_i(t) = -1 + (a+r)p_i(t) \Rightarrow p(t) = Me^{(a+r)t} + \frac{1}{a+r}$$

it does not depend on x_0

$\Rightarrow$ **OLNE subgame perfect**

Linear state game: Not Necessary conditions example

2 Firms producing durable goods.
$s_i(t)$ sales of firm i $\quad$ $X_i(t)$ accumulated sales of firm i up to time t
$\dot{X}_i(t) = s_i(t) = \alpha_i(t)(M - X_1(t) - X_2(t))$
Assuming α_i depend on price strategies $u_i(t)$ and defining the state of the game $x(t) = M - X_1(t) - X_2(t)$

$$\dot{x}(t) = -[\alpha_1(u_1(t), u_2(t)) + \alpha_2(u_1(t), u_2(t))]x(t)$$

Payoffs

$$J_i = \int_0^T e^{-rt}[u_i(t) - c_i] \cdot x(t) - \alpha_i(u_1(t), u_2(t))(M - x_1(t) - x_2(t))\ dt$$

$$\frac{\partial \tilde{H}_i^2}{\partial x^2} = 0 \text{ It is linear state}$$

BUT

$$\frac{\partial \tilde{H}_i^2}{\partial u_i \partial x} \neq 0 \text{ there is a multiplicative term between } u_i \text{ and } x$$

Linear State games

There may be a multiplicative interaction between the state variable of player i, x_i and the control variables of player j, u_j.

And still get a Markov perfect open-loop Nash Equilibrium.

See Example 7.3 Dockner page 191

In Linear State games OLNE $\equiv$ Markov Perfect MNE

Subgame perfect

strong Time consistent

Enjoy with differential games

Enjoy with your research

Find the equilibria of your life!