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Time consistency and Stackelberg games

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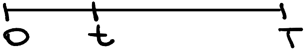
The time consistency issue

(Dockner et al. p.98)

Notation

- Weak time consistency (WTC) $\equiv$ Time consistency (TC)
- Strong time consistency (STC) $\equiv$ Subgame-perfect (SP)

(Weak) time consistency



$\Gamma(0, x_0)$ game from 0 to T, $x(0) = x_0$

$\Gamma(t, x)$ subgame from t to T, $[t, T]$ s.t. $x(t) = x$

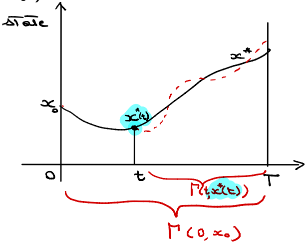
Definition ((Weak) time consistency)

A MNE in $\Gamma(0, x_0)$ *is* weak *time consistent if it is a MNE in any subgame* $\Gamma(t, x)$ *that starts in* $x^*(t)$ $\Rightarrow \Gamma(t, x^*)$

- Any OLNE is (weakly) time consistent
- Any MNE is (weakly) time consistent

$\Phi\big|_{(t,T)}$ s.t.

$x(t) = x^*(t)$



Strong time consistency

Definition ((Strong) time consistency)

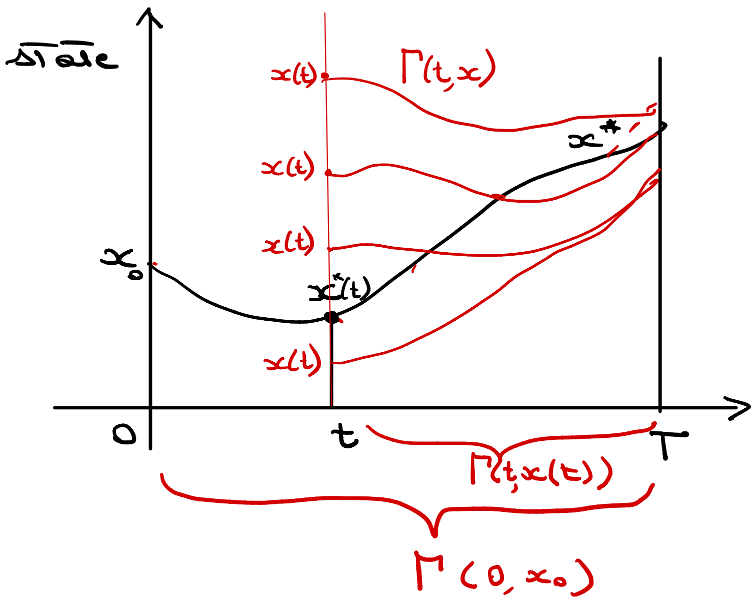
A MNE in $\Gamma(0, x_0)$ *is subgame perfect (strongly time consistent) if it is a MNE in* any *subgame* $\Gamma(t, x), \forall x \in X$ *(either on the optimal equilibrium trajectory OR not). Any* $\Gamma(t, x)$ *is identical to* $\Gamma(0, x_0)$ *except for the initial point.*

Markov Perfect Nash Equilibrium

Theorems

- Any OLNE is NOT subgame perfect (in general)
- Any MNE is subgame perfect
- A MNE with $T = +\infty$ is subgame perfect if ϕ^* is independent of x_0

Subgame perfectness



OLNE NOT subgame perfect: Example

N players

$$\max \quad J^i(u^i()) = -\int_0^T (u^i(t))^2 \, dt - x(T)^2$$
$$\dot{x}(t) = \sum_{j=1}^N u^j(t) = u^1(t) + u^2(t) + \ldots + u^N(t)$$
$$x(0) = 0 = x_0$$
$$u(t) \in \mathcal{R}$$

$J^i(u^i()) \leq 0$ for any feasible control $\Rightarrow$ Optimal value $J^i(u^i()) = 0$

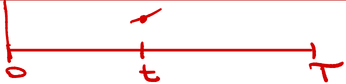
optimal control $u^i(t) \equiv 0 \;\Rightarrow$ Optimal path $x^i(t) \equiv 0$

$x(t) \equiv 0, \quad \Rightarrow$ eq. trajectory $\underline{u^i(t) = \Phi^i(x(t), t) = x(t)}$

$u^{i*}(t)$ is (weak) **Time consistent:** (strategies credible along the eq. trajectory)
Let all players $j \neq i$ use $\Phi^j(x, t) = x$, then player i has to face

$$\begin{cases} \dot{x}(t) = u^i + (N-1)x \\ x(0) = 0 \end{cases} \quad \Rightarrow \quad u^{i*}(t) = 0 \quad \Rightarrow \quad x^*(t) = 0.$$

OLNE NOT subgame perfect: Example



$x(t) \neq x^*(t) = 0$

Strategies not credible along any trajectory

Φ^i not credible as optimal behaviour OFF the equilibrium path

If there exists some time t such that $x(t) \neq 0$, then: $(x>0)$

- All players sticking to Φ^i would have to choose non-zero controls $\Phi^i(x(t), t) = x(t) \neq 0$ state is driven away from 0
- Each player prefers to choose $u^{i*}(t) = 0$ to avoid the cost associated with a non-zero control value and to reduce the speed at which the system diverges from 0.

Although the strategies $\Phi^i(x, t) = x$ are credible along the equilibrium trajectory $x^*(t)$, they are not credible as specifications of optimal behaviour out of the equilibrium path.

HJB $\begin{cases} -V_t = \max_{u_i} \{ \underbrace{-u_i^2 + V_x (\sum_{j=1}^{2} u_j)}_{g(u_i)} \} \\ V(T, x(T)) = -x^2 \end{cases}$ $\longrightarrow$ $g'(u_i) = -2u_i + V_x$

$u_i = \frac{V_x}{2}$

HJB ... $V(x,t) = \cancel{a(t)} + \cancel{b(t)x} + c(t)x^2$

$\Phi_{OLNE} \neq$

$$\boxed{\Phi^i(x,t) = \frac{x}{(2N-1)(t-T)-1}}$$

strong Time consistent feedback (non deg) strategy

$$\boxed{V(x,t) = \frac{x^2}{(2N-1)(t-T)-1}}$$

$\limsup_{t\to+\infty} e^{-rt} V(x_f(t), t) \le 0$ for any x_f feasible trajectory.

$r = 0$, $e^{-rt} = 1$

Markov perfect Nash equilibrium

Stackelberg equilibrium

Sequential, asymmetric information, hierarchical

Leader (L), Follower (F)

a) L: declares his strategy u^L

b) F: computes his best response (rational choice) $u^F = u^F(u^L)$

c) L:

$$\max_{u^L \in \mathcal{U}^{\mathcal{L}}} J^L(u^L, u^F(u^L))$$

OC. problem in u^L

backward induction.

$\underline{(u^L_S, u^F_S)}$ Stackelberg equilibrium

Open-Loop Stackelberg Equilibrium (OLSE)

System dynamics

$$\begin{cases} \dot{x}_i(t) = f_i(x_i(t), u^L(t), u^F(t), t) \\ x_i(0) = x_0 \\ x_i(T) \in \mathcal{R}, u^L(t) \in \mathcal{U}^{\mathcal{L}}, u^F(t) \in \mathcal{U}^{\mathcal{F}} \end{cases}$$

$i = 1, -, m$

a) L: declares his control path $u^L(t)$

b) F: computes his best response

$$\max_{u^F \in \mathcal{U}^{\mathcal{F}}} J^F = \int_0^T \underbrace{e^{-r^F t}} v^F(x(t), u^L, u^F(t), t)\, dt$$

$$H_C^F(x, u^F, \lambda_i, t) = v^F(x, u^L, u^F, t) + \sum_{i=1}^{n} \lambda_i(t) f_i(x, u^L, u^F, t)$$

co-state

concavity, $\mathcal{U}^{\mathcal{F}}$ open, stationary points.

$$\frac{\partial H^F}{\partial u^F} = \frac{\partial v^F(x,u^L,u^F,t)}{\partial u^F} + \sum_{i=1}^{n} \frac{\lambda_i \partial f_i(x,u^L,u^F,t)}{\partial u^F} = 0 \Rightarrow u^F(t)$$

co-state eq.

$$\dot{\lambda}_i(t) = -\frac{\partial H^F}{\partial x_i} = -\frac{\partial v^F(x,u^L,u^F,t)}{\partial x_i} - \sum_{i=1}^{n} \frac{\lambda_i \partial f_i(x,u^L,u^F,t)}{\partial x_i} + r\lambda$$

$+ x\lambda_i$

$$\boxed{\lambda_i(T) = 0}$$

$\exists\ u^F(t) = g(x(t), \lambda(t), u^L(t), t)$ best response of F to the actions of the leader The co-state equation becomes

$$\begin{cases} \dot{\lambda}_i(t) = -\dfrac{\partial v^F(x(t),u^L(t),g(x(t),\lambda(t),u^L(t),t))}{\partial x_i} + \\ \quad - \displaystyle\sum_{i=1}^{n} \frac{\lambda_i \partial f_i(x(t),u^L(t),g(x(t),\lambda(t),u^L(t),t)}{\partial x_i} \\ \lambda_i(T) = 0 \end{cases}$$

is $x(t)$ does not enter $\Rightarrow$ sol. $\tilde{\lambda}_i(t) \Rightarrow \tilde{\lambda}_i(0)$

1. What do we know about $\lambda_i(0)$?
2. Do they depend on the leader's announced time path $u^*(t)$ or not?

The answer depends on the structure of the problem

$\lambda_i(0)$ is CONTROLLED by u^L

$\lambda_i(0)$ is NOT CONTROLLED by u^L

Example 5.1 $\lambda_i(0)$ Controlled by L

$$J^F = \int_0^T \left(u^F - \frac{{u^F}^2}{2} - \frac{x^2}{2} \right) dt$$

$$v^F(x, u^F) = u^F - \frac{(u^F)^2}{2} - \frac{x^2}{2}$$

$$\begin{cases} \dot{x}(t) = u^F(t) + u^L(t) \\ x(0) = x_0 \end{cases}$$

Dynamics

$$H^F(x, u^F, \lambda, t) = u^F - \frac{(u^F)^2}{2} - \frac{x^2}{2} + \lambda(u^F + u^L)$$

$$\frac{\partial H^F}{\partial u^F} = 1 - u^F + \lambda$$

$$\boxed{u^{*F}(t) = 1 + \lambda(t)}$$

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial H^F}{\partial x} = x(t), & \lambda(T) = 0 \\ \dot{x}(t) = (1 + \lambda(t)) + u^L(t) & x(0) = x_0 \end{cases}$$

ODEs are coupled

The Follower's control variable $u^F(t)$ at time t depends also on the future values of $u^L(t)$, i.e. on $u^L(s)$, $s > t$.

Example 5.2 $\lambda_i(0)$ NOT Controlled by L

$$J^F = \int_0^T u^F - \frac{(u^F)^2}{2} - x \, dt$$

$$v^F(x, u^F) = u^F - \frac{(u^F)^2}{2} - x$$

$$\begin{cases} \dot{x}(t) = u^F(t) + u^L(t) \\ x(0) = x_0 \end{cases}$$

$$H^F(x, u^F, \lambda, t) = u^F - \frac{(u^F)^2}{2} - x + \lambda(u^F + u^L)$$

$$\boxed{u^*(t) = 1 + \lambda(t)}$$

co-state

$$* \begin{cases} \dot{\lambda}(t) = -\frac{\partial H^F}{\partial x} = 1 \\ \lambda(T) = 0 \end{cases}$$

independent w.r.t. x

$$\boxed{\lambda(t) = t - T} \rightarrow x(0) = -T$$

State redundant

The Leader has no influence on the follower's best response.

Controllable co-state

Definition

The initial value $\lambda(0)$ of the Follower's co-state function is called

- *Controllable if $\lambda(0)$ depends on $u^L(t)$ (Ex 5.1)*
- *Uncontrollable if $\lambda(0)$ does not depend on $u^L(t)$ (Ex 5.2)*

The Leader's problem

L knows the best response of the Follower

$$\max_{u^L} J^L = \int_0^T e^{-r^L t} v^L(x(t), u^L(t), u^{FBR}(t), t)$$

$\dot{x} =$ —

$\dot{\lambda} =$ —

co-state function of F

$$u^{FBR}(t) = g(x(t), \lambda(t), u^L(t), t), t)$$

The co-state function of F becomes a state function for L $\rightarrow$

additive co-state function π associated with λ

$x(0) = x_0$ fixed

$\lambda_i(0)$

$\lambda(0)$ **is fixed iff it is uncontrollable**

$$\begin{aligned} H_C^L(x, \lambda, u^L, \psi, \pi, t) &= v^L(x, u^L, g(x(t), \lambda(t), u^L(t), t), t) \\ &+ \sum_{i=1}^{n} \psi_i(t) f_i(x, u^L, g(x(t), \lambda(t), u^L(t), t), t), t) + \\ &+ \sum_{i=1}^{n} \pi_i k_i(x, \lambda, u^L, t) \end{aligned}$$

$$\frac{\partial H^L(x(t),\lambda(t),u^L(t),\psi(t),\pi(t),t)}{\partial u^L}=0$$

ψ costate fct associated to x

$$\dot\psi(t)=r^L\pi_i(t)-\frac{\partial H^L(x(t),\lambda(t),u^L(t),\psi(t),\pi(t),t)}{\partial x_i}=$$

π costate fct associated to λ

$$\dot\pi(t)=r^L\pi(t)-\frac{\partial H^L(x(t),\lambda(t),u^L(t),\psi_i(t),\pi(t),t)}{\partial \lambda_i}$$

$$\psi_i(T)=0 \text{ because } x(T)\in\mathcal{R}$$

$$\boxed{\pi_i(0)=?}$$

$$\begin{cases}\text{If } \lambda(0) \text{ is controllable } \Rightarrow & \lambda(0) \text{ treated as a state function of L} \\ & \text{associated co-state } \pi_i(0)=0 \\ \text{If } \lambda(0) \text{ is non-controllable } (\lambda(t)=t-T) \Rightarrow & \text{no need to consider it} \\ & \text{as a state function of L}\end{cases}$$

Non consistent Stackelberg equilibrium

$$
\begin{aligned}
J^L &= \int_0^T u^L(t) - \tfrac{1}{2}\left[(u^L(t))^2 + (x(t))^2\right]\,dt \\
&\dot{x}(t) = 1 + \lambda(t) + u^L(t) \\
&\dot{\lambda}(t) = x(t) \\
&x(0) = 0, \quad x(T) \in \mathcal{R} \\
&\lambda(T) = 0, \quad \lambda(0) \text{ controllable} \quad \Rightarrow \lambda
\end{aligned}
$$

$$H^L(x, \lambda, u^L, \psi, \pi) = u^L - \tfrac{1}{2}(u^L + x^2) + \psi(1 + \lambda + u^L) + \pi x$$

$$
\begin{cases}
1 - u^L(t) + \psi(t) = 0 \\
\dot{\psi}(t) = x(t) - \pi(t) \\
\dot{\pi}(t) = -\psi(t) \\
\psi(T) = 0 \\
\pi(T) = 0
\end{cases}
\qquad z = (\underbrace{x, \lambda}_{\text{state}}, \underbrace{\psi, \pi}_{\text{costate}})
$$

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dot{z} = Bz + k$$

$$\exists!\ SOL$$

At a given time $t_1 > 0$, we have $\pi(t_1) \neq 0$
If L can replan his strategy at the time t_1, he will choose a new solution such that $\pi(t_1) = 0$ (because his co-state fct at t_1 is free) and therefore he will deviate.
The Leader has no longer an incentive to keep his promises.

Consistent Stackelberg equilibrium

(Example 5.2 (continued))

$$\lambda(t) = t - T \quad \lambda(0) = -T$$

$1 + \lambda(t) = 1 + t - T$

Leader's

$$\begin{aligned} J^L &= \int_0^T u^L(t) - \tfrac{1}{2}[(u^L(t))^2 + (x(t))^2]\, dt \\ \dot{x}(t) &= 1 + t - T + u^L(t) \\ x(0) &= 0, \quad x(T) \in \mathcal{R} \end{aligned}$$

$$H^L(x, \lambda, u^L, \psi, \pi) = u^L - \tfrac{1}{2}(u^L + x^2) + \psi(1 + t - T + u^L)$$

$$1 - u^L(t) + \psi(t) = 0 \qquad \Rightarrow u^L(t) = 1 + \psi(t)$$

$$\begin{cases} \dot{x}(t) = \psi(t) + 2 + t - T, & x(0) = 0 \\ \dot{\psi}(t) = x(t), & \psi(T) = 0 \end{cases}$$

... consistent.