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Differential games

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.

DIPARTIMENTO MATEMATICA

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Dynamic game - Players - Information structure

$$[0, T\rangle \text{ Programming interval } T < +\infty, \ T = +\infty$$

N players $i \in \{1, 2, \ldots, N\}$

$u^i(t) : [0, T\rangle \to \mathbb{R}$

$$u^i \in \mathcal{U}^2(x(t), u^{-i}(t), t) \subset \mathcal{R}^{mj} \text{ Player's strategies}$$

Strategies based on the **information** revealed during all times $t \in [0, T\rangle$ when the game takes place.
At any t players have the knowledge of all the previous actions.

Perfect information

Differential game

$\underline{x}(t) : [0, T\rangle \to \mathbb{R}$ state function
The system varies according to a differential equation

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{f}(\underline{x}(t), \underline{u}(t), t) \\ \underline{x}(t_0) &= \underline{x}^0 \\ \underline{x}(t) &\in X \subset \mathcal{R}^2 \end{aligned}$$

Payoffs

discount factor

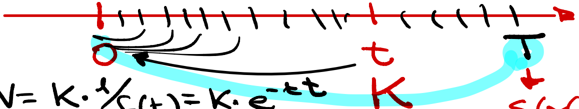
• Scrap Value
• Salvage

$$J^i(u^i(\cdot)) \;=\; \int_0^T e^{-r^i t} F^i(x(t), u^1(t), \ldots, u^N(t), t)\, dt \;+\; e^{-r^i T} S^i(x(T))$$

C

$C \longrightarrow M = C \cdot f(t) = C \cdot e^{rt}$

accumulation factor



$AV = K \cdot 1/f(t) = K \cdot e^{-rt}$

discount factor

$S^i : X \mapsto \mathcal{R}, \qquad T < +\infty$

$S^i(x) = 0, \forall x \qquad T = +\infty$

Problem for each player i

$\forall i \in \{1, \dots, N\}$

(Dockner et al p.86)

$$\max J^i_{\Phi^{-i}}(u^i(\cdot)) = \int_0^T e^{-r^i t} F^i_{\Phi^{-i}}(x(t), u^i(t), t)\, dt + e^{-r^i T} S^i(x(T))$$

$$\text{subject to} \begin{cases} \dot{x}(t) = f^i_{\Phi^{-i}}(x(t), u^i(t), t) \\ x(0) = x^0 \\ u^i(t) \in \mathcal{U}^i_{\Phi^{-i}}(x(t), t) \end{cases}$$

$i \in \{1, \dots, N\}$

$$F^i_{\Phi^{-i}}(x, u^i, t) = F^i(x(t), \Phi^1(\ ), \dots, \Phi^{i-1}(\ ), u^i, \Phi^{i+1}(\ ), \dots, \Phi^N(\ ), t)$$

$f^i_{\Phi^{-i}}$

$\mathcal{U}^i_{\Phi^{-i}}$

player "i" cannot control x on $J^i_{\Phi^{-i}}$

Different types of strategies Φ_i

A) OPEN-LOOP $\Phi_i = \Phi_i(t)$

B) CLOSED-LOOP WITH MEMORY $\Phi_i = \Phi_i(t, x(\tau), 0 \leq \tau \leq t)$ future depends on the past

C) CLOSED-LOOP WITHOUT MEMORY (no-memory) MARKOVIAN STRATEGY $\Phi_i = \Phi_i(t, x(t)), \forall t$ all payoffs depend on the present

   C1) $T = +\infty$: STATIONARY MARKOVIAN (autonomous) $\Phi_i = \Phi_i(x(t))$

Observations

- INFORMATION: Different information is required for the implementation
  Markovian strategy is more demanding from the informative point of view
  If Information is either Irrelevant or Inaccessible $\Longrightarrow$ OPEN-LOOP
- COMMITMENT: Players can deviate from the declared strategy $\Longrightarrow$ MARKOVIAN consistent strategy

Nash equilibrium for a differential game

The $N - tuple$

$$(\Phi^1, \Phi^2, \dots, \Phi^i, \dots, \Phi^N)$$

constitutes a Nash equilibrium iff for all players, $i \in \{1, 2, \dots, N\}$

$$J^i(\Phi^1, \Phi^2, \dots, \Phi^i, \dots, \Phi^N) \geq J^i(\Phi^1, \Phi^2, \dots, u^i, \dots, \Phi^N)$$

$$\forall u^i \in \mathcal{U}^i$$

FIXED Φ^{-i} we need to compute the **best response strategy** of player i

$$\begin{aligned} \max J^i_{\Phi^{-i}}(u^i(\cdot)) &= \int_0^T e^{-r^i t} F^i_{\Phi^{-i}}(x(t), u^i(t), t)\, dt + e^{-r^i T} S^i(x(T)) \\ \text{subject to} \quad \dot{x}(t) &= f^i_{\Phi^{-i}}(x(t), u^i(t), t) \\ x(0) &= x^0 \\ u^i(t) &\in \mathcal{U}^i_{\Phi^{-i}}(x(t), t) \end{aligned}$$

$i \in \{1, \dots, N\}$

Nash equilibrium for a differential game

Finding a **Nash equilibrium** in a differential game with N players is equivalent to **solve** N **Optimal Control problems.**

- OPEN-LOOP NASH EQUILIBRIUM (OLNE)
  Pontryagin's Maximum Principle (1962) PMP

$$(\Phi^1(t), \Phi^2(t), \dots, \Phi^i(t), \dots, \Phi^N(t))$$

- MARKOVIAN NASH EQUILIBRIUM (MNE)
  Hamilton Jacobi Bellman ('50s) HJB

$$(\Phi^1(t, x(t)), \Phi^2(t, x(t)), \dots, \Phi^i(t, x(t)), \dots, \Phi^N(t, x(t)))$$

OPEN-LOOP NASH EQUILIBRIUM (OLNE)

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Nash equilibrium for a differential game

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$$(\Phi^1(t, x(t)), \Phi^2(t, x(t)), \ldots, \Phi^i(t, x(t)), \ldots, \Phi^N(t, x(t)))$$

OPEN-LOOP NASH EQUILIBRIUM (OLNE)

Definition 4.2 The N-tuple $(\phi^1, \phi^2, \dots, \phi^N)$ of functions $\phi^i : [0, T) \mapsto \mathbb{R}^{m^i}$, $i \in \{1, 2, \dots, N\}$, is called an open-loop Nash equilibrium if, for each $i \in \{1, 2, \dots, N\}$, an optimal control path $u^i(\cdot)$ of the problem (4.1) exists and is given by the open-loop strategy $u^i(t) = \phi^i(t)$.

OPEN-LOOP NASH EQ. (OLNE)

Pontryagin Maximum Principle (1962)

Hamiltonian function

$\dot{x}(t) = f^i(\ )$

$$H^i(x, u^I, p, t) = e^{-r^i t} F^i(\ ) + p \cdot f^i(\ )$$

$\to n = \#$ of state functions

$p(t) : [0, T] \mapsto \mathcal{R}^n$ co-state function (piece wise $\mathcal{C}'$)

∃ a co-state fct associated to each state

Current Value Hamiltonian (discount factor is not included in H^c)

Dockner

$$H^{iC}(x, u^I, \lambda, t) = F^i(\ ) + \lambda \cdot f^i(\ )$$

$$H = e^{-rt}F + pf = \underbrace{e^{-rt}}_{} \left(F + \underbrace{e^{rt}p}_{\lambda} f\right) = \underbrace{e^{-rt}}_{>0} H^C$$

$$\operatorname*{argmax}_{u} H \approx \operatorname{argmax} H^C$$

$$\dot{\lambda} = r e^{rt} p + \dots$$

(Dockner p.87)

$$\max_{u\geq 0} J^1(u(\cdot)) = \underbrace{\int_0^T e^{-rt}}\overbrace{\left[v(t) - x(t) - \frac{\alpha}{2}u^2(t)\right]}^{F^1} dt$$

$$\max_{v\in[0,1]} \underbrace{J^2(v(\cdot))} = \int_0^T e^{-rt}\left[v(t) - x(t)\right] dt$$

$$\text{s.t. } \dot{x}(t) = 1 + v(t) - u(t)\sqrt{x(t)},$$

$$x(0) = x^0$$

Assume $v(t) = \psi(t)$ for P2, find best response strategy for P1 $\underline{(P_0 = 1)}$

$H^{C1}(x, u, \lambda, t) = P_0\left(\psi - x - \frac{\alpha}{2}u^2\right) + \lambda\left(1 + \psi - u\sqrt{x}\right)$

Assume $u(t) = \Phi(t)$ for P1, find best response strategy for P2

$H^{C2}(x, v, \lambda, t) = P_0(v - x) + \mu\left(1 + v - \phi\sqrt{x}\right)$ $(P_0 = 1)$

Pontryagin's Maximum Principle approach (discount factor) $\rho \geq 0$

running ~ instantaneous

$$
\begin{aligned}
\text{maximize} \quad J(\underline{u}) \;&=\; \int_0^T e^{-\rho t} F_0(\underline{x}(t), \underline{u}(t), t)\, dt \;+\; e^{-\rho T}\, S(\underline{x}(T)) \\
\text{subject to} \quad \underline{\dot{x}}(t) \;&=\; \underline{f}(\underline{x}(t), \underline{u}(t), t) \\
\underline{x}(0) \;&=\; \underline{x}^0 \\
x_i(T) \;&=\; x_i^1 \quad i = 1, \ldots, l \\
x_i(T) \;&\geq\; x_i^1 \quad i = l+1, \ldots, m \\
x_i(T) \;&\in\; \Re \quad i = m+1, \ldots, n \\
\underline{u}(t) \;&\in\; \Omega
\end{aligned}
$$

$S(x(T), u(T))$

Terminal Cost!

associated **current value** Hamiltonian function

$$H^C(\underline{x}, \underline{u}, \underline{\lambda}, t) = p_0 F_0(x(t), u(t), t) + \underline{\lambda} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

$$H^C(\underline{x}, \underline{u}, \underline{\lambda}, t) = p_0 F_0(x(t), u(t), t) + \sum_{i=1}^{i=n} \lambda_i f_i(\underline{x}(t), \underline{u}(t), t)$$

Pontryagin Maximum Principle (discount factor)

Theorem

Let $u^*(t)$ be a piecewise continuous control defined on $[0, T]$ which solves problem (DOC) and let $x^*(t)$ be the associated optimal path. Then $\exists$ $n+1$ constants $p_0, \gamma_1, \ldots, \gamma_n \in \Re$ and a continuous and piecewise continuously differentiable function $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ such that $\forall t \in [0, T]$

- $(p_0, \gamma_1, \ldots, \gamma_n) \neq (0, 0, \ldots, 0)$
- $u^*(t)$ *maximizes* $H^C(x^*(t), u, \lambda(t), t)$ *for all* $u \in \Omega$
- *Excepts at the points of discontinuities of* $u^*(t)$ *co-state equation (ODE)*

$$\dot{\lambda}_i(t) = -\frac{\partial H^C(x^*(t), u^*(t), \lambda(t), t)}{\partial x_i} + \rho\, \lambda_i(t)$$

- $p_0 \in \{0, 1\}$ *for* $i = 1, \ldots, n$
- *Transversality conditions* ($\rightarrow$ next page)

Theorem

- *Transversality conditions*

$$\lambda_i(T) = p_0 \frac{\partial S(x^*(T))}{\partial x_i} + \gamma_i, \quad i = 1, \dots n$$

where

complementary KKT condition

$$\begin{array}{lll} \gamma_i \in \Re, & i = 1, \dots, l & \text{if } x_i^*(T) = x_i^l \\ \gamma_i \geq 0 & i = l+1, \dots, m & \text{if } x_i^*(T) \geq x_i^l \\ & \gamma_i(x_i^*(T) - x_i^l) = 0 & \\ \gamma_i = 0 & i = m+1, \dots, n & \text{if } x_i^*(T) \in \Re \end{array}$$

Player 1

$$H^{c1}(x,u,\lambda,t) = v - x - \frac{\alpha}{2}u^2 + \lambda(r + v - u\sqrt{x})$$

ii) $u^* \in \underset{u \geq 0}{\arg\max}\, H^{c1}$

$$\frac{\partial H^{c1}}{\partial u} = -\alpha u - \lambda\sqrt{x} = 0 \Rightarrow \hat{u}(t) = -\frac{\lambda(t)\sqrt{x(t)}}{\alpha}$$

$$\frac{\partial^2 H^{c1}}{\partial u^2} = -\alpha < 0$$

H^{c1} concave

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$\mathcal{U}^1 = [0, +\infty)$

iii) COSTATE EQ

$$\dot{\lambda}(t) = -\frac{\partial H^{c1}}{\partial x} + r\lambda(t) = 1 + \frac{\lambda(t)u}{2\sqrt{x}} + r\lambda(t)$$

$$\begin{cases} \dot{\lambda}(t) = 1 - \frac{\lambda^2(t)}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\alpha} + r\lambda(t) \\ \lambda(T) = 0 \end{cases}$$

Riccati eq

$x(T) \in \mathbb{R}$ free $\quad S'(x(T)) = 0$

$$H^{c1}(x,u,\lambda,t) = 0 - x - \frac{\alpha}{2}u^2 + \lambda(1 + 5 - u\sqrt{x})$$

- $u^* = \arg\max_{u \geq 0} H^{c1}$

$$\frac{\partial H^{c1}}{\partial u} = -\alpha u - \lambda\sqrt{x} = 0$$

$$\frac{\partial^2 H^{c1}}{\partial u^2} = -\alpha < 0 \quad \text{concave}$$

$$\boxed{u^*(t) = -\frac{\lambda(t)\sqrt{x(t)}}{\alpha}}$$

- $$\dot\lambda(t) = -\frac{\partial H^{c1}}{\partial x} + r\lambda(t) = 1 + \frac{\lambda(t)u}{2\sqrt{x}} + r\lambda(t)$$

CO-STATE SYSTEM PDE backward

$$\begin{cases} \dot\lambda(t) = 1 - \dfrac{\lambda^2(t)\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}\,\alpha} + r\lambda(t) \\ \lambda(T) = 0 \end{cases}$$

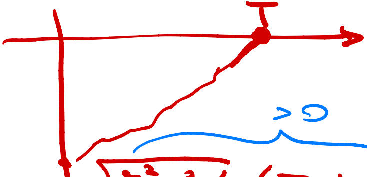
Riccati eq

$$\left|\begin{array}{l} S(x(T)) = 0 \\ x(T) \in \mathbb{R} \end{array}\right. \Rightarrow \gamma = 0$$

Player 1

$\dot{\lambda}(T) = 1$

$\dot{\lambda} > 0$



$$\exists!\ \lambda(t) = \frac{2\left(1 - e^{-\overbrace{\sqrt{x^2+2/\alpha}\ \underbrace{(T-t)}_{>0}}^{>0}}\right)^{>0}}{\underbrace{\underbrace{\left(x-\sqrt{x^2+2/\alpha}\right)e^{-c(T-t)}}_{<(x-\sqrt{\ })<(x+\sqrt{\ })} - \left(x+\sqrt{x^2+2/\alpha}\right)}_{<0}} \le 0$$

$$\Phi(t) = u^* = -\frac{\lambda\sqrt{x}}{\alpha} \ge 0$$

best response

$$\boxed{\lambda(t) = 0 \iff t = T}$$

$$\begin{cases} \dot{x}(t) = 1 + \varphi(t) + \dfrac{\lambda(t)\,x(t)}{\alpha} \\ x(0) = x_0 \end{cases}$$ Linear ODE

$\Rightarrow$ strategy of P2 ?!

$x(t) = \_\_\_\_ \;?$

Such a φ is only a candidate
Apply ARROW's sufficiency Theorem

$$\left(H^{c1}\right)^* = H^{c1}(x, u^*, \lambda, t) = v - x - \frac{\alpha}{2}\left(-\frac{\lambda\sqrt{x}}{\alpha}\right)^2 + \lambda\left(1 + v + \frac{\lambda\sqrt{x}}{\alpha}\sqrt{x}\right)$$

$$= v - x - \frac{\lambda^2 x}{2\alpha} + \lambda + \lambda v + \frac{\lambda^2 x}{\alpha}$$

linear in x
$\Rightarrow$ concave in x
$\Rightarrow$ ARROW ✓

$\Rightarrow \varphi$ is optimal strategy.

$u(t) = \phi(t)$ player 1

$H^{c2}(x, v, \mu, t) = v - x + \mu\,(1 + v - \phi\sqrt{x})$ linear in v

- $\underset{v \in [0,1]}{\operatorname{argmax}} H^{c2} = (1+\mu)v - x + \mu\,(1 - \phi\sqrt{x})$

$$\psi^* = v^* = \begin{cases} 0, & \mu < -1 \\ 1, & \mu > -1 \end{cases}$$ bang-bang sol

$\phi = -\dfrac{\lambda\sqrt{x}}{\alpha}$

- $\dot{\mu}(t) = -\dfrac{\partial H^{c2}}{\partial x} + r\mu(t) = 1 + \dfrac{\mu\,\phi}{2\sqrt{x}} + r\mu(t)$

$$= 1 + r\mu(t) - \frac{\mu\lambda}{2\alpha}$$

- $\mu(T) = 0$

$$\begin{cases} \dot{\lambda} = 1 + r\lambda - \frac{1}{2\alpha}\lambda^2 & \lambda(T) = 0 \\ \dot{\mu} = 1 + r\mu - \frac{1}{2\alpha}\mu\lambda & \mu(T) = 0 \end{cases}$$

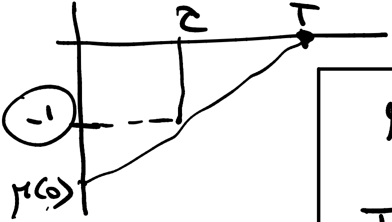
$$\dot{\lambda} - \dot{\mu} = \frac{1}{2\alpha}\lambda(\mu - \lambda) + r(\lambda - \mu)$$

$$y(t) = \lambda - \mu$$

$$\begin{cases} \dot{y}(t) = \frac{\lambda y}{2\alpha} + r y = \underline{\left(\frac{\lambda}{2\alpha} + r\right)} y \Rightarrow y(t) \equiv 0 \\ \underline{y(T) = 0} \end{cases}$$

$$\Downarrow$$

$$\boxed{\lambda(t) = \mu(t)}$$



$$\boxed{\mu(0) < -1 \iff T > \frac{1}{c}\ln\left(\frac{2-\left(x-\sqrt{x^2+2/a}\right)}{2-\left(x+\sqrt{x^2+2/a}\right)}\right) = \tau}$$

$$\mu(0) > -1 \Rightarrow \tau = 0$$

$$\sigma(t) = \psi(t) = \begin{cases} 0, & t \in [0,\tau) \\ 1, & t \in [\tau, T] \end{cases}$$

degenerate strategy not depending on the current state

Player 2

Check optimality

$$H^{*c2}(x, \psi^*, \mu, t) = \psi^* - x + \mu\left(1 + \psi^* - \overset{-\frac{\lambda}{\alpha}\sqrt{x}}{\phi^*}\sqrt{x}\right)$$

0 1   0 1

linear in x $\Rightarrow$ Arrow's Theorem holds

Open-Loop Nash EQ

(ϕ^*, ψ^*)

$$\phi^*(t) = -\frac{\lambda(t)}{\alpha}\sqrt{x(t)}$$

$$\psi^*(t) = \begin{cases} 0 & t \in [0, \tau) \\ 1 & t \in [\tau, T] \end{cases}$$

λ, τ see above

$x^*(t)$ state function

$$\begin{cases} \dot{x}(t) = 1 + v^*(t) - u(t)\sqrt{x(t)} \\ x(0) = x_0 \end{cases}$$

$$u^* = -\frac{\lambda\sqrt{x}}{\alpha}$$

In $[0, \tau)$ $\quad v^* = 0 = \psi^*$

$$\dot{x}(t) = 1 + \frac{\lambda x(t)}{\alpha}$$

$$x(t) = e^{\int_0^t \frac{\lambda(z)}{\alpha} dz} \left\{ x_0 + \int_0^t (1+\psi) e^{-\int_0^s \frac{\lambda(z)}{\alpha} dz} ds \right\}, \quad \psi = 0$$

$$\boxed{x(\tau)} \longrightarrow x_\tau$$

Player 2

In $[\tau, T]$ $\quad \sigma^{2} = 1 = \varphi^{2}$

$$\begin{cases} \dot{x}(t) = 2 + \dfrac{\lambda(t)x(t)}{\alpha} \\ x(\tau) = x_\tau \end{cases}$$

$$x(t) = e^{\int_\tau^t \frac{\lambda(z)}{\alpha} dz} \left[x_\tau + \int_\tau^t 2 e^{-\int_\tau^s \frac{\lambda(z)}{\alpha} dz} ds \right]$$

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