Dipartimento di Matematica "Tullio Levi-Civita"

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# Introduction to differential games

PhD Program in Mathematical Sciences

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# *Course contents (12 hours)*

- Recall of basic concepts of game theory, best response strategies, dominating strategies, Nash equilibrium
- Dynamic games: formalization of a differential game
- Simultaneous Noncooperative differential games (Nash equilibrium)
- Hierarchic differential games (Stackelberg equilibrium)

# *References*


- Basar T., and Olsder G.J., *Dynamic Noncooperative Game Theory* Classics in Applied Mathematics.. SIAM 2 Ed., 1999.
- Dockner, E.J. et al., *Differential Games in Economics and Management Science*, Cambridge University Press, 2000.
- Van Long, N., *A Survey of Dynamic Games in Economics* Surveys on Theories in Economics and Business Administration, Vol. 1, 2010.
- Bressan, A. “Noncooperative differential games.” *Milan Journal of Mathematics* 79.2 (2011) 357-427.
- Jehle, G. A. and Reny P.J., *Advanced Microeconomic Theory* (Third). Essex: Pearson Education Limited, 2011.
- Haurie, A., et al, *Games and dynamic games.* Vol.1 World Scientific Publishing Company, 2012.

# *Exam*

1. The lecturer will suggest a set of recent scientific publications on differential games
2. Each student will choose a paper among the suggested ones to read, comprehend and present in class

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# Introduction to game theory

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# *Game theory*

***Quantitative methods***
***for strategic interactions***
***among entities***

# Our logical thread

| | One player | Many players |
|---|---|---|
| Static | Mathematical programming | (Static) game theory |
| Dynamic | Optimal control theory | Dynamic (and/or differential) game theory |

# *From Mathematical programming to Game Theory*

One decision maker

$\max_{u} J(u),\ u \in U,$ U set of actions

Two decision makers P1 , P2

$\max_{u1} J_1(u1, u2),$ $u1 \in U1$

$\max_{u2} J_2(u1, u2),$ $u2 \in U2$

Game Theory

# *Game*

**Basic elements:**

- **Players** with clear preferences, represented by a **Payoff** function.
- Each **Action** leads to an associated Consequence

***Axioms***:

- Players are rational:

  They are aware of their alternatives, forms expectations about any unknowns, have clear preferences, and choose their action deliberately after some process of optimization.

- And think strategically.

  When designing his strategy for playing the game, each player takes into account any knowledge or expectation he may have regarding his opponents' behaviour.

# *Rational Behavior*

A Set of **Actions** from which the decision-maker makes a choice.

C Set of possible **Consequences** of these actions.

J: A --> C

Consequence function that associates a Consequence with each Action.

Preference relation (a complete transitive reflexive binary relation) on C.

# *Static games (One-shot games)*

- Each player makes one choice and this completely determines the payoffs.
- Zero-Sum (Noncooperative) matrix games ←→ NonZero-sum bimatrix
- Normal (strategic) form: all possible sequences of decisions of each player are set out against each other (no dynamic)
  - Matrix structure

## Normal form

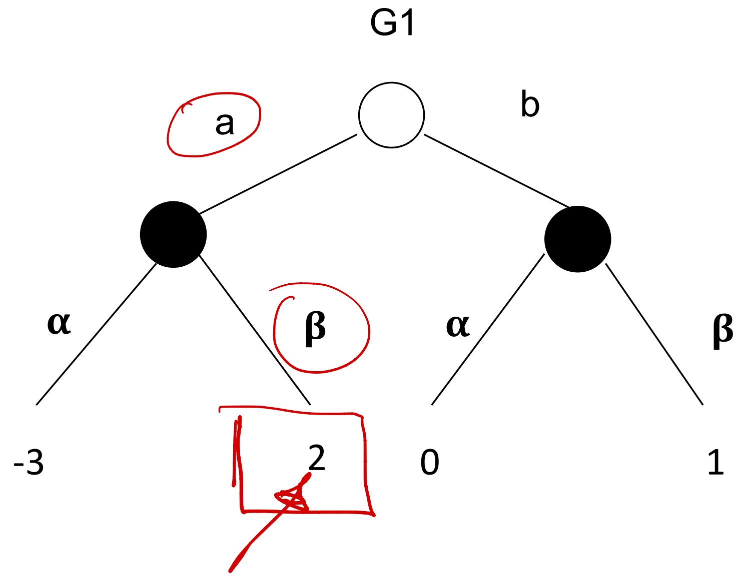
| G1 \ G2 | α | β |
|---|---|---|
| a | (-3, -2) | (2, 0) |
| b | (0, 2) | (1, 1) |

Non-zero sum game

Existence questions

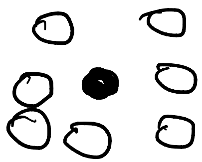
Pure and mixed strategies

## Extensive form for G1




Single-act games

Multi-act games



Go game

## ***Choice of strategies***

WHAT IS OPTIMAL?

# Best response strategies

| P1 \ P2 | α | β |
|---|---|---|
| a | (1,-1) | (0,0) |
| b | (2,-2) | (0,-3) |
| c | (1,-1) | (1,-1) |

$u_i^b$ best reply (response) by player i to a profile of strategies for all other players $u_{-i}$ if

$$J^i(u_i^b, u_{-i}) \geq J^i(u_i, u_{-i}) \text{ for all } u_i \in U^i$$

$u_{-i}$

# *Strictly Dominating strategies*

| P1 \ P2 | α | β |
|---|---|---|
| a | (1,0) | (0,0) |
| b | (2,-2) | (1,0) |
| c | (1,-1) | (0,-1) |

$u_i^d$ of player i

$J^i(u_i^d, u_{-i}) > J^i(u_i, u_{-i})$ for all $u_i \in U^i$,

for all $u_{-i} \in U^1 \times U^2 \times \ldots \times U^{i-1} \times U^{i+1} \times \ldots \times U^N$

# *Dominating strategies*

- Eliminating some rows and/or columns which are known from the beginning to have no influence on the equilibrium solution

- Looking for Saddle points

- best reply to any feasible profile of the $N-1$ rivals:

# Example: Zero Sum Marketing game

Market 1 Market 2

FIRM A 4 units of capital

FIRM B 2 units of capital

**Payoffs of A**

| | | STRATEGIES of **B** | | |
|---|---|---|---|---|
| | | 2 , 0 | 1 , 1 | 0 , 2 |
| STRATEGIES of **A** | 4 , 0 | 1+0=1 | | |
| | 3 , 1 | | | |
| | 2 , 2 | | 1+1=2 | |
| | 1 , 3 | -1+1=0 | | |
| | 0 , 4 | | | |

# *Example: Zero Sum Marketing game -2-*

| A \ B | 2 , 0 | 1 , 1 | 0 , 2 |
|---|---|---|---|
| 4 , 0 | (1,-1) | (0,0) | (0,0) |
| 3 , 1 | (2,-2) | (1,-1) | (0,0) |
| 2 , 2 | (1,-1) | (2,-2) | (1,-1) |
| 1 , 3 | (0,0) | (1,-1) | (2,-2) |
| 0 , 4 | (0,0) | (0,0) | (1,-1) |

# *Dominating strategies Player A*

| A \ B | 2 , 0 | 1 , 1 | 0 , 2 |
|---|---|---|---|
| ~~4 , 0~~ | ~~1~~ | ~~0~~ | ~~0~~ |
| 3 , 1 | 2 | 1 | 0 |
| 2 , 2 | 1 | 2 | 1 |
| 1 , 3 | 0 | 1 | 2 |
| ~~0 , 4~~ | ~~0~~ | ~~0~~ | ~~1~~ |

# *MaxiMin rule*
# *(von Neumann)*

- non-probabilistic decision-making rule
- decisions are ranked on the basis of their worst-case outcomes
- the optimal decision is one with the least worst outcome.

“In the worst of cases…”

# *MaxiMin rule*

| B / A | $s_1^B$ | $s_2^B$ | $s_3^B$ | |
|---|---|---|---|---|
| $s_1^A$ | (7,-7) | (5,-5) | (4,-4) | 4 |
| $s_2^A$ | (2,-2) | (6,-6) | (3,-3) | 2 |
| $s_3^A$ | (8,-8) | (0,0) | (1,-1) | 0 |
| | -8 | -6 | -4 | |

# *MaxiMin rule*

## ***Saddle point***

| A \ B | $s_1^B$ | $s_2^B$ | $s_3^B$ | MIN of A |
|---|---|---|---|---|
| $s_1^A$ | (7,-7) | (5,-5) | (4,-4) | 4 |
| $s_2^A$ | (2,-2) | (6,-6) | (3,-3) | 2 |
| $s_3^A$ | (8,-8) | (0,0) | (1,-1) | 0 |
| MIN of B | -8 | -6 | -4 | |

← MAX MIN of A

↑ MAX MIN of B

# *Saddle points may not exist (∄ )*

G2

| G1 | α | β |
|---|---|---|
| a | (-3, -2) | (2, 0) |
| b | (0, 2) | (1, 1) |

## *Static games in normal form*
## *Choice of strategies*

- Dominating strategies
- MiniMax Theorem (von Newmann)
- but Saddle points existence not guaranteed

# *Nash Equilibrium*

A set of strategies constitutes a Nash equilibrium if no single player in interested in changing his strategy unless one of the other players changes his own.

That is:

Keeping the choices of other players fixed,
Nobody is interested in changing his own.

# *Example with No saddle point but there exists 1 Nash equilibria*

**Set** of strategies (a, **β**) s.t.:

Knowing that G1 playes **a** then for G2 has not choise (convenience)  but to play **β**

Knowing that G2 playes **β** then for G1 has not choise (convenience)  but to play **a**

G2

| G1 | **α** | **β** |
|---|---|---|
| a | (5, 5) | (3, 3) |
| b | (2, 2) | (0, 0) |

# *Example with No saddle point but there exist 2 Nash equilibria*

**Set** of strategies (a, **β**) s.t.:
Knowing that G1 playes **a** then for G2 has not choise (convenience) but to play **β**
Knowing that G2 playes **β** then for G1 has not choise (convenience) but to play **a**

| | | G2 | |
|---|---|---|---|
| | | **α** | **β** |
| G1 | a | (-3, -2) | (2, 0) |
| | b | (0, 2) | (1, 1) |