Optimal control theory
Pontryagin's Maximum Principle

Alessandra Buratto

.

DIPARTIMENTO MATEMATICA

Università degli Studi di Padova

optimal control problem (OC)

$$\begin{aligned}
\text{maximize} \quad & J(\underline{u}) \; = \; \int_{t_0}^{t_1} f_0(\underline{x}(t), \underline{u}(t), t)\, dt \; + \; S(\underline{x}(t_1)) \\
\text{subject to} \quad & \underline{\dot{x}}(t) \; = \underline{f}(\underline{x}(t), \underline{u}(t), t) \\
& \underline{x}(t_0) \; = \; \underline{x}^0 \\
& x_i(t_1) \; = \; x_i^1 \quad i = 1, \ldots, l \\
& x_i(t_1) \; \geq \; x_i^1 \quad i = l+1, \ldots, m \\
& x_i(t_1) \; \in \Re \quad i = m+1, \ldots, n \\
& \underline{u}(t) \; \in \; \Omega \subset \Re^r
\end{aligned}$$

associated Hamiltonian function

$$H(\underline{x}, \underline{u}, \underline{p}, t) = p_0 f_0(x(t), u(t), t) + \underline{p} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

Hamiltonian function

$$H(\underline{x}, \underline{u}, \underline{p}, t) = p_0 f_0(x(t), u(t), t) + \underline{p} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

$$H(\underline{x}, \underline{u}, \underline{p}, t) = p_0 f_0(x(t), u(t), t) + \sum_{i=1}^{i=n} p_i f_i(\underline{x}(t), \underline{u}(t), t)$$

Pontryagin Maximum principle (BOLZA form) 1/2

Theorem

Let $u^*(t)$ be a piecewise continuous control defined on $[t_0, t_1]$ which solves problem (OC) and let $x^*(t)$ be the associated optimal path. Then $\exists\ n+1$ constants $p_0, \gamma_1, \ldots, \gamma_n \in \Re$ and a continuous and piecewise continuously differentiable function $p(t) = (p_1(t), \ldots, p_n(t))$ such that $\forall t \in [t_0, t_1]$

i) $(p_0, \gamma_1, \ldots, \gamma_n) \neq (0, 0, \ldots, 0)$

ii) $u^*(t)$ *maximizes* $H(x^*(t), u, p(t), t)$ *for all* $u \in \Omega$ *that is*

$$H(x^*(t), u^*(t), p(t), t) > H(x^*(t), u, p(t), t) \quad \text{for all } u \in \Omega$$

iii) *Excepts at the points of discontinuities of* $u^*(t)$ *for* $i = 1, 2, \ldots, n$

$$\dot{p}_i(t) = -\frac{\partial H(x^*(t), u^*(t), p(t), t)}{\partial x_i} - - - - >$$

Pontryagin Maximum principle (BOLZA form) 2/2

Theorem

iv) $p_0 \in \{0, 1\}$

v) *Transversality conditions*

$$p_i(t_1) = p_0 \frac{\partial S(x^*(t_1))}{\partial x_i} + \gamma_i, \quad i = 1, \dots n$$

where

$$\begin{array}{lll} \gamma_i \in \Re, & i = 1, \dots, l & \text{if } x_i^*(t_1) = x_i^l \\ \gamma_i \geq 0 & i = l+1, \dots, m & \text{if } x_i^*(t_1) \geq x_i^l \\ & \gamma_i(x_i^*(t_1) - x_i^l) = 0 & \\ \gamma_i = 0 & i = m+1, \dots, n & \text{if } x_i^*(t_1) \in \Re \end{array}$$

Transversality conditions - Examples

Transversality conditions - Other formulations

$$x_i(t_1) \;\leq\; x_i^1$$

v) Transversality conditions

$$p_i(t_1) = p_0 \frac{\partial S(x^*(t_1))}{\partial x_i} - \gamma_i, \quad i = 1, \ldots n$$

Exercise 1

$$
\begin{aligned}
\text{maximize} \quad J(u) &= \int_0^2 x(t)\,dt \\
\text{subject to} \quad \dot{x}(t) &= u(t) - x(t) \\
x(0) &= x^0 \\
\\
u(t) &\in [0,1]
\end{aligned}
$$

Exercize 2

$$\begin{aligned}
\text{maximize} \quad J(u) &= \int_0^1 \left(x(t) - \frac{u^2(t)}{2} \right) dt \\
\text{subject to} \quad \dot{x}(t) &= u(t) - x(t) \\
x(0) &= 0 \\
x(1) &\in \Re \\
u(t) &\in \Re
\end{aligned}$$

- Ex 2 with $u(t) \in [0, 1]$

- Ex 2 with $u(t) \in [0, 1/2]$

Exercize 3

$$
\begin{aligned}
\textbf{minimize} \quad & J(u) = -\int_0^1 x(t)\, dt \\
\text{subject to} \quad & \dot{x}(t) = u(t) + x(t) \\
& x(0) = 0 \\
& x(1) = 1 \\
& u(t) \in [-1, 1]
\end{aligned}
$$

Exercize 4

$$
\begin{aligned}
\textbf{minimize} \quad J(u) &= \int_0^3 -\frac{u^2(t)}{2}\, dt \\
\text{subject to} \quad \dot{x}(t) &= u(t) - x(t)/3 \\
x(0) &= 0 \\
x(3) &\geq 10 \\
u(t) &\geq 0
\end{aligned}
$$

Exercize 5

$$\begin{array}{rrl} \text{maximize} & J(u) & = -\frac{1}{2} \int_0^T u^2(t)\,dt + \quad x(T) \\ \text{subject to} & \dot{x}(t) & = u(t) + M(1 - x(t)) \qquad M > 0 \\ & x(0) & = \; 0 \end{array}$$

Optimal control problem discount factor (DOC)

$$
\begin{aligned}
\text{maximize} \quad & J(\underline{u}) = \int_0^T e^{-\rho t} F_0(\underline{x}(t), \underline{u}(t), t)\, dt + e^{-\rho T} S(\underline{x}(T)) \\
\text{subject to} \quad & \dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t) \\
& \underline{x}(0) = \underline{x}^0 \\
& x_i(T) = x_i^1 \quad i = 1, \dots, l \\
& x_i(T) \geq x_i^1 \quad i = l+1, \dots, m \\
& x_i(T) \in \Re \quad i = m+1, \dots, n \\
& \underline{u}(t) \in \Omega
\end{aligned}
$$

associated **current value** Hamiltonian function

$$
H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \underline{q} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)
$$

$$
H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \sum_{i=1}^{i=n} q_i f_i(\underline{x}(t), \underline{u}(t), t)
$$

Pontryagin Maximum principle (discount factor)

Theorem

Let $u^*(t)$ be a piecewise continuous control defined on $[0, T]$ which solves problem (DOC) and let $x^*(t)$ be the associated optimal path. Then $\exists$ $n+1$ constants $p_0, \gamma_1, \ldots, \gamma_n \in \Re$ and a continuous and piecewise continuously differentiable function $q(t) = (q_1(t), \ldots, q_n(t))$ such that $\forall t \in [0, T]$

- $(p_0, \gamma_1, \ldots, \gamma_n) \neq (0, 0, \ldots, 0)$
- *$u^*(t)$ maximizes $H^C(x^*(t), u, q(t), t)$ for all $u \in \Omega$*
- *Excepts at the points of discontinuities of $u^*(t)$*

$$\dot{q}_i(t) = -\frac{\partial H^C(x^*(t), u^*(t), q(t), t)}{\partial x_i} + \rho\, q(t)$$

- $p_0 \in \{0, 1\}$ *for $i = 1, \ldots, n$*
- *Transversality conditions* *($\rightarrow$ next page)*

$+$

Theorem

- *Transversality conditions*

$$q_i(T) = p_0 \frac{\partial S(x^*(T))}{\partial x_i} + \gamma_i, \quad i = 1, \dots n$$

where

$$\begin{array}{lll} \gamma_i \in \Re, & i = 1, \dots, l & \text{if } x_i^*(T) = x_i^l \\ \gamma_i \geq 0 & i = l+1, \dots, m & \text{if } x_i^*(T) \geq x_i^l \\ & \gamma_i(x_i^*(T) - x_i^l) = 0 & \\ \gamma_i = 0 & i = m+1, \dots, n & \text{if } x_i^*(T) \in \Re \end{array}$$

Exercise

$$
\begin{aligned}
\text{maximize} \quad J(u) &= \int_0^1 \frac{u^2(t)}{2} e^{-\rho t}\, dt \\
\text{subject to} \quad \dot{x}(t) &= u(t) - x(t)/10 \\
x(0) &= 5 \\
x(1) &\geq 10 \\
u(t) &\geq 0
\end{aligned}
$$

Sufficient conditions

Theorem (Mangasarian sufficient conditions)

Let $(x^*(t), u^*(t))$ be an admissible pair. Suppose Ω convex and $\frac{\partial f_i}{\partial u_j}$ exist continuous. If there exist a continuous and piecewise cont. differentiable function $p(t)$ such that the following conditions are satisfied with $p_0 = 1$,

- $\dot{p}_i(t) = -\frac{\partial H(x^*(t), u^*(t), p(t), t)}{\partial x_i}$
- $u^*(t)$ *maximizes* $H(x^*(t), u, p(t), t)$ *for all* $u \in \Omega$
- $\gamma_i \in \Re, \quad i = 1, \ldots, l$ if $x_i^*(t) = x_i'$

  $\gamma_i \geq 0 \quad \gamma_i(x_i^*(t) - x_i') = 0 \quad i = l+1, \ldots, m$ if $x_i^*(t) \geq x_i'$

  $\gamma_i = 0 \quad i = m+1, \ldots, n$ if $x_i^*(t) \in \Re$
- $H(x, u, p(t), t)$ *is* concave in (x, u) *for all* t.

then $(x^*(t), u^*(t))$ *is the optimal solution of* (OC).

Mangasarian sufficient conditions - uniqueness

H strictly concave in $(x, u), \forall t$

$$\Downarrow$$

$(x^*(t), u^*(t))$ is unique

Sufficient conditions

Theorem (Arrow sufficient conditions)

Let $(x^*(t), u^*(t))$ be an admissible pair. If there exist a continuous and piecewise cont. differentiable function $p(t)$ such that the following conditions are satisfied with $p_0 = 1$,

- $\dot{p}_i(t) = -\frac{\partial H(x^*(t), u^*(t), p(t), t)}{\partial x_i}$
- $u^*(t)$ *maximizes* $H(x^*(t), u, p(t), t)$ *for all* $u \in \Omega$
- $\gamma_i \in \Re, \qquad i = 1, \ldots, l \quad \text{if } x_i^*(t) = x_i^l$
  $\gamma_i \geq 0 \quad \gamma_i(x_i^*(t) - x_i^l) \qquad i = l+1, \ldots, m \text{ if } x_i^*(t) \geq x_i^l$
  $\gamma_i = 0 \qquad i = m+1, \ldots, n \text{ if } x_i^*(t) \in \Re$
- $\hat{H}(x, p(t), t) = \max_{u \in \Omega} H(x, u, p(t), t) = H(x, u^*, p(t), t)$ *is* concave in x *for all* t.

then $(x^*(t), u^*(t))$ is the optimal solution of (OC).

Arrow sufficient conditions- uniqueness

$$\hat{H} \text{ strictly concave in } x, \forall t$$

$$\Downarrow$$

$$(x^*(t), u^*(t)) \text{ is unique}$$

Optimal control problem infinite horizon

$$\begin{aligned} \text{maximize} \quad J(\underline{u}) &= \int_{t_0}^{+\infty} e^{-\rho t} F_0(\underline{x}(t), \underline{u}(t), t)\, dt \\ \text{subject to} \quad \dot{\underline{x}}(t) &= \underline{f}(\underline{x}(t), \underline{u}(t), t) \\ \underline{x}(t_0) &= \underline{x}^0 \\ \underline{u}(t) &\in \Omega \end{aligned}$$

$$H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \underline{q} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

Necessary conditions: Pontryagin Maximum Principle **without transversality conditions**

Sufficient conditions:

$$\lim \inf_{T \to +\infty} e^{-\rho T} q(T)(x(T) - x^*(T)) \geq 0 \text{ for all feasible} x(t)$$

Steady state solutions

Definition

A dynamic system is in a steady state if the state variable associated to the optimal (steady state) control are unchanging in time.

In this context, the system is in a steady state if

$$\underline{\dot{x}}^{SS}(t) \;= \underline{f}(\underline{x}^{SS}(t), \underline{u}^{SS}(t), t) = \underline{0}$$

We study the equilibria of a dynamic system