# Differential games

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# Dynamic game - Players - Information structure

$$[0, T\rangle \text{ Programming interval } t < +\infty,\ t = +\infty$$

$N$ players $i \in \{1, 2, \ldots, N\}$

$$u^i \in \mathcal{U}^2(x(t), u^{-i}(t), t) \subset \mathcal{R}^{mj} \text{ Player's strategies}$$

Strategies based on the **information** revealed during all times $t \in [0, T\rangle$ when the game takes place.
At any $t$ players have the knowledge of all the previous actions.

**Perfect information**

# Differential game

The system varies according to a differential equation

$$
\begin{aligned}
\underline{\dot{x}}(t) &= \underline{f}(\underline{x}(t), \underline{u}(t), t) \\
\underline{x}(t_0) &= \underline{x}^0 \\
\underline{x}(t) &\in X \subset \mathcal{R}^2
\end{aligned}
$$

# Payoffs

$$J^i(u^i(\cdot)) \;=\; \int_0^T e^{-r^i t} F^i(x(t), u^1(t), \ldots, u^N(t), t)\, dt \;+ e^{-r^i T}\; S^i(x(T))$$

$S^i : X \mapsto \mathcal{R}, \quad T < +\infty$
$S^i(x) = 0, \forall x \quad T = +\infty$

# Problem for each player $i$

(Dockner et al p.86)

$$\begin{aligned} \max J^i_{\Phi^{-i}}(u^i(\cdot)) &= \int_0^T e^{-r^i t} F^i_{\Phi^{-i}}(x(t), u^i(t), t)\, dt \ + e^{-r^i T}\ S^i(x(T)) \\ \text{subject to} \quad \dot{x}(t) &= f^i_{\Phi^{-i}}(x(t), u^i(t), t) \\ x(0) &= x^0 \\ u^i(t) &\in \mathcal{U}^i_{\Phi^{-i}}(x(t), t) \end{aligned}$$

$i \in \{1, \ldots, N\}$

$$F^i_{\Phi^{-i}}(x, u^i, t) = F^i(x(t), \Phi^1(\ ), \ldots, \Phi^{i-1}(\ ), u^i, \Phi^{i+1}(\ ), \ldots, \Phi^N(\ ), t)$$

$f^i_{\Phi^{-i}}$

$\mathcal{U}^i_{\Phi^{-i}}$

# Different types of strategies $\Phi_i$

A) OPEN-LOOP $\Phi_i = \Phi_i(t)$

B) CLOSED-LOOP WITH MEMORY $\Phi_i = \Phi_i(t, x(i), 0 \leq i \leq t)$ future depends on the past

C) CLOSED-LOOP WITHOUT MEMORY (no-memory) MARKOVIAN STRATEGY $\Phi_i = \Phi_i(t, x(T)), \forall t$ all payoffs depend on the present

  C1) STATIONARY MARKOVIAN $\Phi_i = \Phi_i(t, x(T))$ $T = +\infty$

**Observations**

- INFORMATION: Different information is required for the implementation
  Markovian strategy is informationally more demanding
  If Information is either Irrelevant or Inacessible $\Longrightarrow$ OPEN-LOOP
- COMMITMENT: Players can deviate from the declared strategy $\Longrightarrow$ MARKOVIAN

# Nash equilibrium for a differential game

The $N - tuple$

$$(\Phi^1, \Phi^2, \dots, \Phi^i, \dots, \Phi^N)$$

constitutes a Nash equilibrium iff

For all players, $i \in \{1, 2, \dots, N\}$

$$J^i(\Phi^1, \Phi^2, \dots, \Phi^i, \dots, \Phi^N) \geq J^i(\Phi^1, \Phi^2, \dots, u^i, \dots, \Phi^N)$$

$$\forall u^i \in \mathcal{U}^i$$

Finding a Nash equilibrium in a differential game with $N$ players is equivalent to solve $N$ Optimal Control problems.

- OPEN-LOOP NASH EQUILIBRIUM (OLNE)
- MARKOVIAN NASH EQUILIBRIUM (MNE)

# OPEN-LOOP NASH EQUILIBRIUM (OLNE)

**Definition 4.2** The $N$-tuple $(\phi^1, \phi^2, \ldots, \phi^N)$ of functions $\phi^i : [0, T) \mapsto \mathbb{R}^{m^i}$, $i \in \{1, 2, \ldots, N\}$, is called an open-loop Nash equilibrium if, for each $i \in \{1, 2, \ldots, N\}$, an optimal control path $u^i(\cdot)$ of the problem (4.1) exists and is given by the open-loop strategy $u^i(t) = \phi^i(t)$.

## OPEN-LOOP NASH EQ. (OLNE)

Pontryagin Maximum Principle (1962)

Hamiltonian function

$$H^I(x, u^I, \lambda, t) = e^{-r^i t} \, F^i(\ ) + \lambda \cdot f^i(\ )$$

$\lambda(t) : [0, t] \mapsto \mathcal{R}^n$

Current Value Hamiltonian

$$H^{iC}(x, u^I, \lambda, t) = \ F^i(\ ) + \lambda \cdot f^i(\ )$$

# MARKOVIAN NASH EQUILIBRIUM (MNE)

**Definition 4.1** The $N$-tuple $(\phi^1, \phi^2, \ldots, \phi^N)$ of functions $\phi^i : X \times [0, T) \mapsto \mathbb{R}^{m^i}$, $i \in \{1, 2, \ldots, N\}$, is called a Markovian Nash equilibrium if, for each $i \in \{1, 2, \ldots, N\}$, an optimal control path $u^i(\cdot)$ of the problem (4.1) exists and is given by the Markovian strategy $u^i(t) = \phi^i(x(t), t)$.

## MARKOVIAN NASH EQ. (MNE)

Hamilton Jacobi Bellman Equation approach

Assume existence of Value function $V^i(x, t) = \max J^i$

# Pontryagin Maximum Principle approach

$$\begin{aligned}
\text{maximize} \quad J(\underline{u}) &= \int_0^T e^{-\rho t} F_0(\underline{x}(t), \underline{u}(t), t)\, dt \; + e^{-\rho T}\; S(\underline{x}(T)) \\
\text{subject to} \quad \dot{\underline{x}}(t) &= \underline{f}(\underline{x}(t), \underline{u}(t), t) \\
\underline{x}(0) &= \underline{x}^0 \\
x_i(T) &= x_i^1 \quad i = 1, \dots, l \\
x_i(T) &\geq x_i^1 \quad i = l+1, \dots, m \\
x_i(T) &\in \Re \quad i = m+1, \dots, n \\
\underline{u}(t) &\in \Omega
\end{aligned}$$

associated **current value** Hamiltonian function

$$H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \underline{q} \cdot \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

$$H^C(\underline{x}, \underline{u}, \underline{q}, t) = p_0 F_0(x(t), u(t), t) + \sum_{i=1}^{i=n} q_i f_i(\underline{x}(t), \underline{u}(t), t)$$

# Pontryagin Maximum Principle (discount factor)

**Theorem**

*Let $u^*(t)$ be a piecewise continuous control defined on $[0, T]$ which solves problem (DOC) and let $x^*(t)$ be the associated optimal path. Then $\exists$ $n+1$ constants $p_0, \gamma_1, \ldots, \gamma_n \in \Re$ and a continuous and piecewise continuously differentiable function $q(t) = (q_1(t), \ldots, q_n(t))$ such that $\forall t \in [0, T]$*

- $(p_0, \gamma_1, \ldots, \gamma_n) \neq (0, 0, \ldots, 0)$
- $u^*(t)$ *maximizes* $H^C(x^*(t), u, q(t), t)$ *for all* $u \in \Omega$
- *Excepts at the points of discontinuities of* $u^*(t)$

$$\dot{q}_i(t) = -\frac{\partial H^C(x^*(t), u^*(t), q(t), t)}{\partial x_i} + \rho\, q(t)$$

- $p_0 \in \{0, 1\}$ *for* $i = 1, \ldots, n$
- *Transversality conditions* *(→ next page)*

## Theorem

- *Transversality conditions*

$$q_i(T) = p_0 \frac{\partial S(x^*(T))}{\partial x_i} + \gamma_i, \quad i = 1, \ldots n$$

*where*

$$\begin{array}{lll} \gamma_i \in \Re, & i = 1, \ldots, l & \text{if } x_i^*(T) = x_i^l \\ \gamma_i \geq 0 & i = l+1, \ldots, m & \text{if } x_i^*(T) \geq x_i^l \\ & \gamma_i(x_i^*(T) - x_i^l) = 0 & \\ \gamma_i = 0 & i = m+1, \ldots, n & \text{if } x_i^*(T) \in \Re \end{array}$$

# Hamilton Jacobi Bellman Equation approach

$$\text{maximize} \quad \int_{t_0}^{T} e^{-r(t-t_0)} F_0\left(x(t), u\left(t\right), t\right) dt + e^{-r(T-t_0)} S(x(T))$$

$$\begin{aligned} \text{subject to} \quad \dot{x}\left(t\right) &= f\left(x\left(t\right), u\left(t\right), t\right) \\ x\left(t_0\right) &= x \\ u\left(t\right) &\in \mathcal{U}(x(t), t) \end{aligned}$$

Looking for feedback strategies $\Phi(x(t), t)$

Value function $V\left(x, t\right)$

# Hamilton Jacobi Bellman Equation approach: Value function

## Definition

*If $P_{x,t}$ has an optimal solution, then let* $V(x,t)$ *be the optimal value*

$$V(x,t) = \max_{u(\cdot)} \left\{ \int_t^T e^{-r(s-t)} F_0(x(s;u), u(s), s)\, ds + e^{-r(T-t)} S(X(T;u)) \right.$$

*where $x(s;u)$ is the unique solution to*

$$\begin{cases} \dot{x}(s) = f(x(s), u(s), s) \\ X(t) = x \end{cases}$$

*associated with the control $u(\cdot)$*

$$V(x,T) = S(x)$$

We assume the existence of the Value Function $V(x,t)$

# Differentiability of the Value Function

The differentiability of the Value Function is not assured, even if $F_0, f \in \mathcal{C}^\infty$

$$\begin{aligned} \text{maximize} \quad & \int_t^T x(s)u(s)dt \\ \text{subject to} \quad & \dot{x}(s) \;=\; u(s) \\ & x(t) = x \\ & u(s) \;\in\; [-1, 1] \end{aligned}$$

$V(x, t) = \frac{(T-t)^2}{2} + (T - t)|x|$

NOT differentiable in$x = 0$ for all $t < T$.

# Sufficiency conditions: Hamilton Jacobi Bellman Equation

## Theorem (HJB equation)

*Let $V : X \times [0, T] \mapsto \mathcal{R}$ be a continuously differentiable function which satisfies the HJB equation*

$$rV(x,t) - \frac{\partial V(x,t)}{\partial t} = \max_{u \in \mathcal{U}(x,t)} \left\{ F_0(x,u,t) + \frac{\partial V(x,t)}{\partial x} f(x,u,t) \right\} \tag{1}$$

*and the terminal condition*

$$V(x,T) = S(x)$$

*for all $(x,t) \in X \times [0,T]$. Let $\Phi(x,t)$ denote the set of controls $u \in \mathcal{U}(x,t)$ maximizing the RHS of* **??**. *If $u(\cdot)$ is a feasible control path with corresponding state trajectory $x(\cdot)$ and if $u(t) \in \Phi(x(t),t)$ holds for almost all $t \in [0,T]$ then $u(\cdot)$ is an optimal control path. Moreover $V(x,t)$ is the optimal value (function) of problem $P_{x,t}$.*

# Hamilton Jacobi Bellman approach

a) Write HJB Eq assuming $V(x,t)$ differentiable

$$rV(x,t) - \frac{\partial V(x,t)}{\partial t} = \max_{u \in \mathcal{U}} \left\{ F_0(x,u,t) + \frac{\partial V(x,t)}{\partial x} f(x,u,t) \right\}$$

b) Find from RHS $\max_{u \in \mathcal{U}} \{ \ \} \Rightarrow u^*(x,t,V_x) \ \forall t, \forall x$

c) Insert $u^*$ in HJB eq. $\Rightarrow$ PDE

$$rV(x,t) - V_x(x,t) = F_0(x,u^*,t) + V_x(x,t) f(x,u^*,t)$$
$$V(x,T) = S(x(T)) = S(x)$$

d) From c) try to guess a form for $V(x,t)$ (generally a polynomial with degree equal to the maximum degree between HJB and Boudary condition (see example))