DIPARTIMENTO MATEMATICA
Dipartimento di Matematica "Tullio Levi-Civita"

Introduction to differential games

PhD Program in Mathematical Sciences

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Course contents (12 hours)

- Recall of basic concepts of game theory, best response strategies, dominating strategies, Nash equilibrium
- Dynamic games: formalization of a differential game
- Simultaneous Noncooperative differential games (Nash equilibrium)
- Hierarchic differential games (Stackelberg equilibrium)

References

- Basar T., and Olsder G.J., *Dynamic Noncooperative Game Theory* Classics in Applied Mathematics.. SIAM 2 Ed., 1999.
- Dockner, E.J. et al., *Differential Games in Economics and Management Science*, Cambridge University Press, 2000.
- Van Long, N., *A Survey of Dynamic Games in Economics* Surveys on Theories in Economics and Business Administration, Vol. 1, 2010.
- Bressan, A. "Noncooperative differential games." *Milan Journal of Mathematics* 79.2 (2011) 357-427.
- Jehle, G. A. and Reny P.J., *Advanced Microeconomic Theory* (Third). Essex: Pearson Education Limited, 2011.
- Haurie, A., et al, *Games and dynamic games.* Vol.1 World Scientific Publishing Company, 2012.

Exam

1. The lecturer will suggest a set of recent scientific publications on differential games
2. Each student will choose a paper among the suggested ones to read, comprehend and present in class

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Introduction to game theory

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Game theory

Quantitative methods
for strategic interactions
among entities

Motivations

MILITARY	**Gulf war,…**
ECONOMICS - MARKETING	**Advertising, Promotion, Price, …**
ECONOMICS – FINANCE	**Portfolio Management**
POLITICS	**Voting systems,…**
SPORT	**Attack / Defense Strategies**
SOCIOLOGY	**Migration, …**
MEDICINE- BIOLOGY	**Neurons, Bacterial evolution**
PSICOLOGY	**Prisoners' dilemma, …**
ENVIRONMENT	**Pollution, Kyoto cartel, …**

… LOGIC – PHILOSOPHY– RELIGION …

A little bit of history

1928	von Neumann	Minimax Theorem
1940	von Neumann,Turing, Zu	Computer → MILITARY
1944	von Neumann, Morgenstern	The Theory of Games and Economic Behavior → ECONOMICS
1950	Nash	Equilibrium & Bargaining
1951	Isaacs	Differential games
1953	Nash, Gillies, Shapley	Threat Core Value
1957	Bellmann	Dynamic programming (DP)
1962	Pontryagin	Pontryagin ‘s Maximum Principle (OC)

Nobel prizes in Economics

1994	John F. Nash Jr. John Harsanyi Reinhard Seltens	PERFECT EQUILIBRIUM	Non cooperative
2005	Y. Robert J. Aumann Thomas C. Schelling	COOPERATION & CONFLICT	
2007	Roger Myerson Leonid Hurwicz Eric Maskin	MECHANISM DESIGN	
2012	Lloyd Shapley Alvin Roth	MARKET DESIGN & STABLE ALLOCATIONS	Sargent Sims 2011
2014	Jean Tirole	MARKET REGULATIONS	

Our logical thread

	One player	Many players
Static	Mathematical programming	(Static) game theory
Dynamic	Optimal control theory	Dynamic (and/or differential) game theory

From Mathematical programming to Game Theory

One decision maker

$\max_{u} J(u),\ u \in U,$ U set of actions

Two decision makers P1 , P2

$\max_{u1} J(u1, u2),$ $u1 \in U1$

$\max_{u2} J(u1, u2),$ $u2 \in U2$

Game Theory

Game

Basic elements:

- **Players** with clear preferences, represented by a **Payoff** function.
- Each **Action** leads to an associated Consequence

Axioms:

- Players are rational:

  They are aware of their alternatives, forms expectations about any unknowns, have clear preferences, and choose their action deliberately after some process of optimization.

- And think strategically.

  When designing his strategy for playing the game, each player takes into account any knowledge or expectation he may have regarding his opponents' behaviour.

Rational Behavior

A Set of **Actions** from which the decision-maker makes a choice.

C Set of possible **Consequences** of these actions.

J: A --> C

Consequence function that associates a Consequence with each Action.

Preference relation (a complete transitive reflexive binary relation) on C.

Static games (One-shot games)

- Each player makes one choice and this completely determines the payoffs.
- Zero-Sum (Noncooperative) matrix games ←→ NonZero-sum bimatrix
- Normal (strategic) form: all possible sequences of decisions of each player are set out against each other (no dynamic)
  - Matrix structure

Normal form

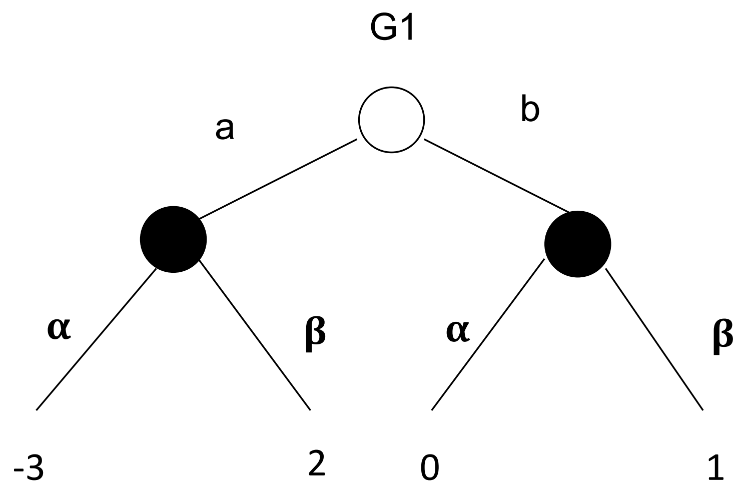
G2

G1		α	β
	a	(-3, -2)	(2, 0)
	b	(0, 2)	(1, 1)

Existence questions

Pure and mixed strategies

Extensive form for G1



Single-act games

Multi-act games

Choice of strategies

WHAT IS OPTIMAL?

Best response strategies

P1 \ P2	α	β
a	(1,-1)	(0,0)
b	(2,-2)	(0,-3)
c	(1,-1)	(1,-1)

u_i^b best reply (response) by player 1 to a profile of strategies for all other players u_{-i} if

$$J^i(u_i^b , u_{-i}) \geq J^i(u_i , u_{-i}) \text{ for all } u_i \in U^i$$

Strictly Dominating strategies

P1 \ P2	α	β
a	(1,0)	(0,0)
b	(2,-2)	(1,0)
c	(1,-1)	(0,-1)

u_i^d of player I

$J^i(u_i^d, u_{-i}) > J^i(u_i, u_{-i})$ for all $u_i \in U^i$,

for all $u_{-i} \in U^1 \times U^2 \times \ldots \times U^{i-1} \times U^{i+1} \times \ldots \times U^N$

Dominating strategies

- Eliminating some rows and/or columns which are known from the beginning to have no influence on the equilibrium solution

- Looking for Saddle points

- best reply to any feasible profile of the $N - 1$ rivals:

Example: Zero Sum Marketing game

M1 Market

M2 Market

FIRM A 4 units of capital

FIRM B 2 units of capital

Payoffs of A

STRATEGIES of **A**	STRATEGIES of **B**		
	2 , 0	1 , 1	0 , 2
4 , 0	1+0=1		
3 , 1			
2 , 2		1+1=2	
1 , 3	-1+1=0		
0 , 4			

Example: Zero Sum Marketing game -2-

TV VE

A \ B	2 , 0	1 , 1	0 , 2
4 , 0	(1,-1)	(0,0)	(0,0)
3 , 1	(2,-2)	(1,-1)	(0,0)
2 , 2	(1,-1)	(2,-2)	(1,-1)
1 , 3	(0,0)	(1,-1)	(2,-2)
0 , 4	(0,0)	(0,0)	(1,-1)

Dominating strategies Player A

A \ B	2 , 0	1 , 1	0 , 2
~~4 , 0~~	~~1~~	~~0~~	~~0~~
3 , 1	2	1	0
2 , 2	1	2	1
1 , 3	0	1	2
~~0 , 4~~	~~0~~	~~0~~	~~1~~

MaxiMin rule (von Neumann)

- non-probabilistic decision-making rule
- decisions are ranked on the basis of their worst-case outcomes
- the optimal decision is one with the least worst outcome.

"In the worst of cases…"

MaxiMin rule

B / A	s_1^B	s_2^B	s_3^B
s_1^A	(7,-7)	(5,-5)	(4,-4)
s_2^A	(2,-2)	(6,-6)	(3,-3)
s_3^A	(8,-8)	(0,0)	(1,-1)

MaxiMin rule

Saddle point

B / A	s_1^B	s_2^B	s_3^B	MIN of A
s_1^A	(7,-7)	(5,-5)	(4,-4)	4
s_2^A	(2,-2)	(6,-6)	(3,-3)	2
s_3^A	(8,-8)	(0,0)	(1,-1)	0
MIN of B	-8	-6	-4	

← MAX MIN of A

↑ MAX MIN of B

Saddle points may not exist (∄)

G2

G1		**α**	**β**
	a	(-3, -2)	(2, 0)
	b	(0, 2)	(1, 1)

Static games in normal form
Choice of strategies

- Dominating strategies
- MiniMax Theorem (von Newmann)
- Saddle points existence not guaranteed

Nash Equilibrium

A set of strategies constitutes a Nash equilibrium if no single player in interested in changing his strategy unless one of the other players changes his own.

That is:

Keeping the choices of other players fixed,
Nobody is interested in changing his own.

Example with No saddle point but there exists 1 Nash equilibria

Set of strategies (a, **β**) s.t.:
Knowing that G1 playes **a** then for G2 has not choise (convenience) but to play **β**
Knowing that G2 playes **β** then for G1 has not choise (convenience) but to play **a**

G2

G1		**α**	**β**
	a	(5, 5)	(3, 3)
	b	(2, 2)	(0, 0)

Example with No saddle point but there exist 2 Nash equilibria

Set of strategies (a, **β**) s.t.:
Knowing that G1 playes **a** then for G2 has not choise (convenience) but to play **β**
Knowing that G2 playes **β** then for G1 has not choise (convenience) but to play **a**

		G2	
		α	**β**
G1	a	(-3, -2)	(2, 0)
	b	(0, 2)	(1, 1)

Nash equilibria for static games

Existence of Nash equilibrium Kakutani fixed point theorem for multivalued maps. Consequence of the classical Brouwer fixed point theorem.

In a zero-sum game, if a Nash equilibrium exists, then all Nash equilibria yield the same payoff V (value of the game)
(von Neumann)

Nash Equilibrium Existence Theorem (1950)

In a finite game there exists **at least one** Nash equilibrium (eventually **mixed strategies**)

P2

P1		y1	y2
	z1	3	0
	z2	-1	1

(u_i^N , u_{-i}^N) Nash equilibrium

$$J^i (u_i^N , u_{-i}^N) > J^i(u_i , u_{-i}^N) \text{ for all } u_i \in U^i$$

Nash Equilibrium

- There might be other combination of strategies that increase the payoff of some players without reducing the payoffs of the others. Or, more, that increase the payoff of all players: **Prisoners’ dilemma**.

Prisoners' Dilemma

- If **only one** confesses, **and puts the blame on the other one**, then he is set free and the other will be sentenced to 6 years of jail;
- If **both** confess, they will be sentenced to 5 years.
- If **neither one** confesses, they will be sentenced to 1 year.

	C	NC		Min A
C	(-5 , -5)	(0 , -6)	-5	←
NC	(-6, 0)	(-1 , -1)	-6	

Prisoners' Dilemma A

	C	NC	Min A
C	(-5 , -5)	(0 , -6)	-5
NC	(-6, 0)	(-1 , -1)	-6

Prisoners' Dilemma B

	C	*NC*
C	(-5 , -5)	(0 , -6)
NC	(-6, 0)	(-1 , -1)
Min B	-5	-6

↑

Prisoners' Dilemma

	A	*NA*	
A	(-5 , -5)	(0 , -6)	← MaxMin A
NA	(-6, 0)	(-1 , -1)	← Cooperative solution

↑ Max Min of B

Prisoners' Dilemma
Nash equilibrium

	A	*NA*
A	(-5 , -5)	(0 , -6)
NA	(-6, 0)	(-1 , -1)

Nash Equilibrium

- Existence and uniqueness is not guaranteed
  → There might exist more that one NE

- It gives solution when there might be uncertainty
- Each player does what is better for him (noncooperative)
- It might not be the better solution for everybody.
- Someone might increase his payoff moving far from the equilibrium.

Nash Equilibrium might not be Pareto Optimum.

Nash equilibrium

Noncooperative simultaneous game

- Symmetric Information structure

$$\max_{u_1 \in U^1} J_1(u_1,u_2) \Rightarrow u_1^{BR}=u_1(u_2)$$

$$\max_{u_2 \in U^2} J_2(u_1,u_2) \Rightarrow u_2^{BR}=u_2(u_1)$$

$$\left.\begin{array}{l} u_1^{BR}=u_1(u_2) \\ u_2^{BR}=u_2(u_1) \end{array}\right\} (u_1^N,u_2^N)$$

Stackelberg game

Noncooperative sequential game

- Asymmetric information structure

1. LEADER: declares his action u_L
2. FOLLOWER: computes his best response $u_F(u_L)$ (to any Leader's strategy u_L)
3. LEADER: computes his optimal Stackelberg strategy u_L^S
4. FOLLOWER: adjust his strategy to obtain the Stackelberg strategy u_F^S

$$\mathit{Max}\ J_F(u_L, u_F) \quad \Rightarrow \quad u_F^{BR} = u_F(u_L) \quad \Rightarrow \quad \mathit{Max}\ J_L(u_L, u_F(u_L))$$

$u_F \in U^F$ $\qquad$ $u_L \in U^L$

$$(u_L^S, u_F^S)$$

Coordination game

Cooperative simultaneous game

- Symmetric information structure

$$\mathit{Max}\ J_1(u_1,u_2)+J_2(u_1,u_2)$$

$$u_1,u_2 \in U^1 \text{ X } U^2$$

Example Cournot duopoly
static game with infinite strategy sets

$J_1=(\alpha-\beta(Q_1+Q_2))Q_1-K_1Q_1^2$ $\quad$ $J_2=(\alpha-\beta(Q_1+Q_2))Q_2-K_2Q_2^2$

NASH: $(Q_1^N,Q_2^N)=\left(\frac{\alpha}{2\,K_1+3\beta},\frac{\alpha}{2\,K_2+3\beta}\right)$

Symm.case $\alpha=\beta=1, K_i=0 \;\Rightarrow (Q_1^N,Q_2^N)=(1/3,1/3)$. $J_1^N= J_2^N=1/9$

STACKELBERG: $(QLS,QFS)=\left(\frac{\alpha\left(1-\frac{\beta}{2(K_F+\beta)}\right)}{2\,(K_L+\beta)-\beta^2/(K_F+\beta)},\frac{\alpha-\beta Q_L{}^S}{2\,(K_F+\beta)}\right)$

Symm. case $\alpha=\beta=1, K_i=0 \;\Rightarrow (Q_1^S,Q_2^S)=(1/2,1/4)$ $J_L^S=1/8$, $J_F^S=1/16$

COOPERATIVE: Symm. case $\alpha=\beta=1, K_i=0 \;\Rightarrow\;$ $J^C=2/9 = J_1^N+ J_2^N$

IN GENERAL $(\alpha,\beta \in \mathbb{R}, K_i=0) \;\Rightarrow\;$ $J^C>J_1^N+ J_2^N$

$J^C > J_1^N+ J_2^N$