CALCULUS 1
Information engineering

02-13-2023

THEME 1

Exercise 1 (8 points) Consider the function

$$f(x) = x^2(\log|x| - 4)$$

(a) determine the maximal domain of f, the sign of f, and possible simmetries (i.e. even or odd);

(b) find limits, points where the function can be continuously prolonged, and possible asymptotes;

(c) Study the differentiability of f, compute the derivative (and possible limits of the derivative); discuss the monotonicity of f and find, if existing, infimums,supremums, maximums (relative and absolute), and minimums(relative and absolute);

(d) plot a qualitative graph of f

Exercise 2 (8 points) In $\mathbb{C}$ find the solutions of the following equation and express them in algebraic form (i.e., like $x + iy$):

$$\sqrt{3}z^2 - 2z - i = 0.$$

Exercise 3 (8 points) Given the sequence

$$a_n = \frac{1}{\sin\left(\frac{1}{n^a}\right)}\left[1 + \frac{1}{3n} - \left(1 + \frac{1}{n}\right)^{1/3}\right],$$

(a) determine, for every value of the parameter $a > 0$, a number $\beta \in \mathbb{R}$ such that $(a_n)_n$ is asymptotic to $\frac{1}{n^\beta}$;

(b) for every value of the parameter $a > 0$, determine the character of the series $\sum_{n=1}^{+\infty} a_n$.

(a)

$$a_n = \frac{1}{\sin\left(\frac{1}{n^a}\right)}\left[1 + \frac{1}{3n} - \left(1 + \frac{1}{n}\right)^{1/3}\right] = \frac{1 + \frac{1}{3n} - 1 - \frac{1}{3n} + \frac{1}{9n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} =$$

$$= \frac{\frac{1}{9n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} \sim \frac{1}{9n^{2-a}}$$

so $\beta = 2 - a$.

(b) Since the sequence (a_n) (has positive terms and) is asymptotic to the sequence $\frac{1}{9n^{2-a}}$, by the asymptotic comparison principle the series $\sum_{n=1}^{+\infty} a_n$ converges if and only if the series $\sum_{n=1}^{+\infty} \frac{1}{9n^{a-2}}$ converges, which happens if and only if $2 - a > 1$, i.e. $a < 1$. Being a series with positive terms, $\sum_{n=1}^{+\infty} a_n$ diverges for every $a \geq 1$.

Exercise 4 (8 points) Consider the function

$$f_\alpha(x) = \frac{1}{(1-x)x^\alpha} \qquad (\alpha \in \mathbb{R}).$$

(a) Compute the generalized integral

$$\int_0^{1/4} f_{1/2}(x)\,dx$$

(b) For every value of the parameter $\alpha \in \mathbb{R}$, study the convergence of the generalized integral

$$\int_0^{1/4} f_\alpha(x)\,dx.$$

(a)

$$\int_0^{1/4} f_{1/2}(x)\,dx = \lim_{c\to 0+} \int_c^{1/4} \frac{1}{(1-x)x^{\frac{1}{2}}}dx \underset{y=x^{\frac{1}{2}}}{=} \lim_{c\to 0+} \int_{c^{\frac{1}{2}}}^{1/2} \frac{2y}{(1-y^2)y}dy =$$

$$\lim_{c\to 0+} \int_{c^{\frac{1}{2}}}^{1/2} \frac{2}{(1-y^2)}dy = \lim_{c\to 0+} \int_{c^{\frac{1}{2}}}^{1/2} \left(\frac{1}{1+y} + \frac{1}{1-y}\right) dy$$

$$\lim_{c\to 0+} \left(\log(1+1/2) - \log(1+c^{\frac{1}{2}}) - \log(1-1/2) + \log(1-c^{\frac{1}{2}})\right) =$$

$$= \log(1+1/2) - \log(1-1/2) = \log(3)$$

because

$$\frac{2}{(1-y^2)} = \frac{A}{(1+y)} + \frac{B}{(1-y)} = \frac{(A+B)+(B-A)y}{(1-y^2)} \qquad \forall y \in \mathbb{R} \iff A = B = 1$$

(b) For $x \to 0$ one has that $f_\alpha \sim \frac{1}{x^\alpha}$; therefore the integral

$$\int_0^{1/4} f_\alpha(x)\,dx.$$

converges if and only if $\alpha < 1$

Taylor expansions.

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n} x^n + o(x^n) \qquad \forall n \geq 0$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \qquad \forall n \geq 0$$

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THEME 2

Exercise 1 (8 points) Consider the function

$$f(x) = x^2(2\log|x| - 3)$$

(a) determine the maximal domain of f, the sign of f, and possible simmetries (i.e. even or odd);

(b) find limits, points where the function can be continuously prolonged, and possible asymptotes;

(c) Study the differentiability of f, compute the derivative (and possible limits of the derivative); discuss the monotonicity of f and find, if existing, infimums,supremums, maximums (relative and absolute), and minimums(relative and absolute);

(d) plot a qualitative graph of f

Exercise 2 (8 points) In $\mathbb{C}$ find the solutions of the following equation and express them in algebraic form (i.e., like $x + iy$):

$$\sqrt{3}z^2 - 2z + i = 0.$$

Exercise 3 (8 points) Given the sequence

$$a_n = \frac{1}{\tan\left(\frac{1}{n^a}\right)}\left[\left(1+\frac{2}{n}\right)^{-1/2} - 1 + \frac{1}{n}\right],$$

(a) determine, for every value of the parameter $a > 0$, a number $\beta \in \mathbb{R}$ such that $(a_n)_n$ is asymptotic to $\frac{1}{n^\beta}$;

(b) for every value of the parameter $a > 0$, determine the character of the series $\sum_{n=1}^{+\infty} a_n$.

(a)

$$a_n = \frac{1}{\tan\left(\frac{1}{n^a}\right)}\left[\left(1+\frac{2}{n}\right)^{-1/2} - 1 + \frac{1}{n}\right] = \frac{1 - \frac{1}{n} + \frac{3}{2n^2} - 1 + \frac{1}{n} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} =$$

$$= \frac{\frac{3}{2n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} \sim \frac{3}{2n^{2-a}}$$

so $\beta = 2 - a$.

(b) Since the sequence (a_n) (has definitely positive terms and) is asymptotic to the sequence $\frac{3}{2n^{2-a}}$, by the asymptotic comparison principle the series $\sum_{n=1}^{+\infty} a_n$ converges if and only if the series $\sum_{n=1}^{+\infty} \frac{3}{2n^{2-a}}$ converges, which happens if and only if $2 - a > 1$, i.e. $a < 1$. Being a series with positive terms, $\sum_{n=1}^{+\infty} a_n$ diverges for every $a \geq 1$.

Exercise 4 (8 points) Consider the function

$$f_\alpha(x) = \frac{1}{(9-x)x^\alpha} \qquad (\alpha \in \mathbb{R}).$$

(a) Compute the generalized integral

$$\int_0^1 f_{1/2}(x)\,dx$$

(b) For every value of the parameter $\alpha \in \mathbb{R}$, study the convergence of the generalized integral

$$\int_0^1 f_\alpha(x)\,dx.$$

(a)

$$\int_0^1 f_{1/2}(x)\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{(9-x)x^{\frac{1}{2}}}dx \underset{y=x^{\frac{1}{2}}}{=} \lim_{c\to 0+} \int_{c^{1/2}}^1 \frac{2y}{(9-y^2)y}dy =$$

$$\lim_{c\to 0+} \int_{c^{1/2}}^1 \frac{2}{(9-y^2)}dy = \lim_{c\to 0+} \int_{c^{1/2}}^1 \frac{1}{3}\left(\frac{1}{3+y} + \frac{1}{3-y}\right)dy$$

$$\lim_{c^{1/2}\to 0+} \frac{1}{3}\left(\log(1+3) - \log(3+c^{\frac{1}{2}}) - \log(3-1) + \log(3-c^{\frac{1}{2}})\right) =$$

$$= \frac{1}{3}\log(4/3) - \frac{1}{3}\log(2/3) = \frac{1}{3}\log(2)$$

because

$$\frac{2}{(9-y^2)} = \frac{A}{(3+y)} + \frac{B}{(3-y)} = \frac{3(A+B)+3(B-A)y}{(1-y^2)} \qquad \forall y \in \mathbb{R} \iff A = B = \frac{1}{3}$$

(b) For $x \to 0$ one has that $f_\alpha \sim \frac{1}{x^\alpha}$; therefore the integral

$$\int_0^1 f_\alpha(x)\,dx.$$

converges if and only if $\alpha < 1$

Taylor expansions.

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + o(x^n) \qquad \forall n \geq 0$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6) \qquad \forall n \geq 0$$

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THEME 3

Exercise 1 (8 points) Consider the function

$$f(x) = x^2(3\log|x| - 2)$$

(a) determine the maximal domain of f, the sign of f, and possible simmetries (i.e. even or odd);

(b) find limits, points where the function can be continuously prolonged, and possible asymptotes;

(c) Study the differentiability of f, compute the derivative (and possible limits of the derivative); discuss the monotonicity of f and find, if existing, infimums,supremums, maximums (relative and absolute), and minimums(relative and absolute); (d) plot a qualitative graph of f

Exercise 2 (8 points) In $\mathbb{C}$ find the solutions of the following equation and express them in algebraic form (i.e., like $x + iy$):

$$\sqrt{3}z^2 + 2z - i = 0.$$

Exercise 3 (8 points) Given the sequence

$$a_n = \frac{1}{\sinh\left(\frac{1}{n^a}\right)}\left[\left(1+\frac{1}{n}\right)^{-1/3} - 1 + \frac{1}{3n}\right],$$

(a) determine, for every value of the parameter $a > 0$, a number $\beta \in \mathbb{R}$ such that $(a_n)_n$ is asymptotic to $\frac{1}{n^\beta}$;

(b) for every value of the parameter $a > 0$, determine the character of the series $\sum_{n=1}^{+\infty} a_n$.

(a)

$$a_n = \frac{1}{\sinh\left(\frac{1}{n^a}\right)}\left[\left(1+\frac{1}{n}\right)^{-1/3} - 1 + \frac{1}{3n}\right] = \frac{1 - \frac{1}{3n} + \frac{2}{9n^2} - 1 + \frac{1}{3n} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} =$$

$$= \frac{\frac{2}{9n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} \sim \frac{2}{9n^{2-a}}$$

so $\beta = 2 - a$.

(b) Since the sequence (a_n) (has definitely positive terms and) is asymptotic to the sequence $\frac{2}{9n^{2-a}}$, by the asymptotic comparison principle the series $\sum_{n=1}^{+\infty} a_n$ converges if and only if the series $\sum_{n=1}^{+\infty} \frac{2}{9n^{2-a}}$ converges, which happens if and only if $2 - a > 1$, i.e. $a < 1$. Being a series with positive terms, $\sum_{n=1}^{+\infty} a_n$ diverges for every $a \geq 1$.

Exercise 4 (8 points) Consider the function

$$f_\alpha(x) = \frac{1}{(4-x)x^\alpha} \qquad (\alpha \in \mathbb{R}).$$

(a) Compute the generalized integral

$$\int_0^1 f_{1/2}(x)\, dx$$

(b) For every value of the parameter $\alpha \in \mathbb{R}$, study the convergence of the generalized integral

$$\int_0^1 f_\alpha(x)\, dx.$$

bigskip (a)

$$\int_0^1 f_{1/2}(x)\, dx = \lim_{c\to 0+} \int_c^1 \frac{1}{(4-x)x^{\frac{1}{2}}} dx \underset{y=x^{\frac{1}{2}}}{=} \lim_{c\to 0+} \int_{c^{1/2}}^1 \frac{2y}{(4-y^2)y} dy =$$

$$\lim_{c\to 0+} \int_{c^{1/2}}^1 \frac{2}{(4-y^2)} dy = \lim_{c\to 0+} \frac{1}{2} \int_{c^{1/2}}^1 \left(\frac{1}{2+y} + \frac{1}{2-y} \right) dy =$$

$$= \lim_{c^{1/2}\to 0+} \frac{1}{2} \left(\log(2+1) - \log(2+c^{\frac{1}{2}}) - \log(2-1) + \log(2-c^{\frac{1}{2}}) \right) = \frac{1}{2}\log 3$$

because

$$\frac{2}{(4-y^2)} = \frac{A}{(2+y)} + \frac{B}{(2-y)} = \frac{2(A+B)+2(B-A)y}{(4-y^2)} \qquad \forall y \in \mathbb{R} \iff A = B = \frac{1}{2}$$

(b) For $x \to 0$ one has that $f_\alpha \sim \frac{1}{x^\alpha}$; therefore the integral

$$\int_0^1 f_\alpha(x)\, dx.$$

converges if and only if $\alpha < 1$

Taylor expansions.

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n} x^n + o(x^n) \qquad \forall n \geq 0$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + + \frac{1}{5!}x^5 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + o(x^{2n+2}) \qquad \forall n \geq 0$$

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THEME 4

Exercise 1 (8 points) Consider the function

$$f(x) = x^2(4\log|x| - 1)$$

(a) determine the maximal domain of f, the sign of f, and possible simmetries (i.e. even or odd);

(b) find limits, points where the function can be continuously prolonged, and possible asymptotes;

(c) Study the differentiability of f, compute the derivative (and possible limits of the derivative); discuss the monotonicity of f and find, if existing, infimums,supremums, maximums (relative and absolute), and minimums(relative and absolute);

(d) plot a qualitative graph of f

Exercise 2 (8 points) In $\mathbb{C}$ find the solutions of the following equation and express them in algebraic form (i.e., like $x + iy$):

$$\sqrt{3}z^2 + 2z + i = 0.$$

Exercise 3 (8 points) Given the sequence

$$a_n = \frac{1}{\arctan\left(\frac{1}{n^a}\right)}\left[1 + \frac{1}{n} - \left(1 + \frac{2}{n}\right)^{1/2}\right],$$

(a) determine, for every value of the parameter $a > 0$, a number $\beta \in \mathbb{R}$ such that $(a_n)_n$ is asymptotic to $\frac{1}{n^\beta}$;

(b) for every value of the parameter $a > 0$, determine the character of the series $\sum_{n=1}^{+\infty} a_n$.

(a)

$$a_n = \frac{1}{\arctan\left(\frac{1}{n^a}\right)}\left[1 + \frac{1}{n} - \left(1 + \frac{2}{n}\right)^{1/2}\right] = \frac{1 + \frac{1}{n} - 1 - \frac{1}{n} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} =$$

$$= \frac{\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^a} + o\left(\frac{1}{n^a}\right)} \sim \frac{1}{2n^{2-a}}$$

so $\beta = 2 - a$.

(b) Since the sequence (a_n) (has definitely positive terms and) is asymptotic to the sequence $\frac{1}{2n^{2-a}}$, by the asymptotic comparison principle the series $\sum_{n=1}^{+\infty} a_n$ converges if and only if the series $\sum_{n=1}^{+\infty} \frac{1}{2n^{2-a}}$ converges, which happens if and only if $2 - a > 1$, i.e. $a < 1$. Being a series with positive terms, $\sum_{n=1}^{+\infty} a_n$ diverges for every $a \geq 1$.

Exercise 4 (8 points) Consider the function

$$f_\alpha(x) = \frac{1}{(16 - x)x^\alpha} \qquad (\alpha \in \mathbb{R}).$$

(a) Compute the generalized integral

$$\int_0^1 f_{1/2}(x)\,dx$$

(b) For every value of the parameter $\alpha \in \mathbb{R}$, study the convergence of the generalized integral

$$\int_0^1 f_\alpha(x)\,dx.$$

bigskip (a)

$$\int_0^1 f_{1/2}(x)\,dx = \lim_{c\to 0+}\int_c^1 \frac{1}{(16-x)x^{\frac{1}{2}}}dx \underset{y=x^{\frac{1}{2}}}{=} \lim_{c\to 0+}\int_{c^{1/2}}^1 \frac{2y}{(4-y^2)y}dy =$$

$$\lim_{c\to 0+}\int_{c^{1/2}}^1 \frac{2}{(16-y^2)}dy = \lim_{c\to 0+}\frac{1}{4}\int_{c^{1/2}}^1 \left(\frac{1}{4+y}+\frac{1}{4-y}\right)dy$$

$$\lim_{c^{1/2}\to 0+}\frac{1}{4}\left(\log(1+4)-\log(4+c^{\frac{1}{2}})-\log(4-1)+\log(4-c^{\frac{1}{2}})\right) =$$

$$= \frac{1}{4}\log(5/4) - \frac{1}{4}\log(3/4) = \frac{1}{4}\log(5/3)$$

because

$$\frac{2}{(16-y^2)} = \frac{A}{(4+y)}+\frac{B}{(4-y)} = \frac{4(A+B)+4(B-A)y}{(4-y^2)} \qquad \forall y\in\mathbb{R} \iff A=B=\frac{1}{4}$$

(b) For $x\to 0$ one has that $f_\alpha \sim \frac{1}{x^\alpha}$; therefore the integral

$$\int_0^1 f_\alpha(x)\,dx.$$

converges if and only if $\alpha < 1$

Taylor expansions.

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \left(\begin{array}{c} a \\ n \end{array}\right)x^n + o(x^n) \qquad \forall n\geq 0$$

$$\arctan(x) = 1 - \frac{x}{3} + \frac{x^5}{5} - \frac{x^7}{7} + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \qquad \forall n\geq 0$$