Lesson 10 - Bosons in a double-well potential
Unit 10.2 Josephson effect

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Two-site Bose-Hubbard Hamiltonian (I)

We have seen that, in the case of a symmetric double-well potential, the two-site Bose-Hubbard Hamiltonian is given by

$$\hat{H} = -J\left(\hat{a}_L^+\hat{a}_R + \hat{a}_R^+\hat{a}_L\right) + \frac{U}{2}\left[\hat{N}_L(\hat{N}_L - 1) + \hat{N}_R(\hat{N}_R - 1)\right] , \quad (1)$$

where J is the hopping (or tunneling) energy while U is the on-site interaction energy.
The Heisenberg equation of motion of the operator $\hat{a}_j$ is given by

$$i\hbar\frac{d}{dt}\hat{a}_j = [\hat{a}_j, \hat{H}] = \frac{\partial\hat{H}}{\partial\hat{a}_j^+} , \quad (2)$$

from which we obtain

$$i\hbar\frac{d}{dt}\hat{a}_j = -J\hat{a}_i + U\hat{N}_j\hat{a}_j , \quad (3)$$

where $j = L, R$ and $i = R, L$.

Time-dependent coherent states (I)

By averaging the Heisenberg equation of motion with the coherent state

$$|\alpha_L \alpha_R\rangle = |\alpha_L\rangle \otimes |\alpha_R\rangle \tag{4}$$

such that

$$\hat{a}_j(t)|\alpha_j\rangle = \alpha_j(t)|\alpha_j\rangle \ , \tag{5}$$

we find

$$i\hbar \frac{d}{dt}\alpha_j = -J\alpha_i + U|\alpha_j|^2 \alpha_j \ , \tag{6}$$

where

$$\alpha_j(t) = \sqrt{\bar{N}_j(t)}\, e^{i\theta_j(t)} \ , \tag{7}$$

with $\bar{N}_j(t)$ the average number of bosons in the site j at time t and $\theta_j(t)$ the corresponding phase angle at the same time t.

Time-dependent coherent states (II)

Working with a fixed number of bosons, i.e.

$$N = \bar{N}_L(t) + \bar{N}_R(t) \ , \tag{8}$$

and introducing population imbalance

$$z(t) = \frac{\bar{N}_L(t) - \bar{N}_R(t)}{N} \tag{9}$$

and phase difference

$$\theta(t) = \theta_R(t) - \theta_L(t) \ , \tag{10}$$

the time-dependent equations for $\alpha_L(t)$ and $\alpha_R(t)$ can be re-written as follows

$$\frac{dz}{dt} = -\frac{2J}{\hbar}\sqrt{1-z^2}\,\sin(\theta) \ , \tag{11}$$

$$\frac{d\theta}{dt} = \frac{2J}{\hbar}\frac{z}{\sqrt{1-z^2}}\cos(\theta) + \frac{UN}{\hbar}\,z \ . \tag{12}$$

These are the so-called Josephson equations of the macroscopic quantum tunneling.

Josephson equations and Josephson effect (I)

The Josephson equations

$$\frac{dz}{dt} = -\frac{2J}{\hbar}\sqrt{1-z^2}\,\sin(\theta)\;, \qquad (13)$$

$$\frac{d\theta}{dt} = \frac{2J}{\hbar}\frac{z}{\sqrt{1-z^2}}\cos(\theta) + \frac{UN}{\hbar}\,z \qquad (14)$$

describe the dynamics of the population imbalance $z(t)$ and relative phase $\theta(t)$ of identical bosons which are performing a macroscopic quantum tunneling.

Quite remarkably, under the condition of small population imbalance ($|z| \ll 1$) one finds

$$\frac{dz}{dt} = -\frac{2J}{\hbar}\sin(\theta)\;, \qquad (15)$$

$$\frac{d\theta}{dt} = \left(\frac{2J}{\hbar}\cos(\theta) + \frac{UN}{\hbar}\right) z\;. \qquad (16)$$

These equations were introduced in 1962 by Brian Josephson to describe the superconducting electric current (made of quasi-bosonic Cooper pairs of electrons) between two superconductors separated by a thin insulating barrier.

DC Josephson effect (I)

If the population imbalance $z(t)$ is extremely small, from the Josephson equations one finds

$$\frac{dz}{dt} = -\frac{2J}{\hbar} \sin(\theta) , \quad (17)$$

$$\frac{d\theta}{dt} = 0 , \quad (18)$$

and consequently

$$\frac{dz(t)}{dt} = -\frac{2J}{\hbar} \sin(\theta(0)) . \quad (19)$$

This is the direct current (DC) Josephson effect: an initial phase difference $\theta(0)$ induces a continuous energy current $\mathcal{J} = -\hbar \frac{dz}{dt}$ of particles, given by

$$\mathcal{J} = \mathcal{J}_{max} \sin(\theta(0)) , \quad (20)$$

through the double-well barrier, with $\mathcal{J}_{max} = 2J$.

AC Josephson effect (I)

If the population imbalance $z(t)$ is small (but not extremely small) and the relative phase $\theta(t)$ is small, one finds instead

$$\frac{dz}{dt} = -\frac{2J}{\hbar}\,\theta \, , \tag{21}$$

$$\frac{d\theta}{dt} = \left(\frac{2J}{\hbar} + \frac{UN}{\hbar}\right) z \, , \tag{22}$$

and consequently

$$\frac{d^2 z}{dt^2} + \frac{2J}{\hbar}\left(\frac{2J}{\hbar} + \frac{UN}{\hbar}\right) z = 0 \, . \tag{23}$$

This is the alternating current (AC) Josephson effect: there is a periodic oscillation of the population imbalance between the two wells of the double-well potential with frequency

$$\Omega = \frac{1}{\hbar}\sqrt{2J\,(2J + UN)} \, . \tag{24}$$