Lesson 7 - Superfluids
Unit 7.2 Equations of superfluid hydrodynamics

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Madelung transformation (I)

Assuming a large number N of particles and the normalization condition

$$N = \int d^3\mathbf{r}\, |\psi_0(\mathbf{r},t)|^2 \qquad (1)$$

for the wavefunction $\psi_0(\mathbf{r},t)$, the time-dependent Gross-Pitaevskii equation can be written as

$$i\hbar \frac{\partial}{\partial t}\psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + g\,|\psi_0(\mathbf{r},t)|^2 \right] \psi_0(\mathbf{r},t) \; . \qquad (2)$$

Adopting the Madelung transformation, namely setting

$$\psi_0(\mathbf{r},t) = n(\mathbf{r},t)^{1/2}\, e^{i\theta(\mathbf{r},t)} \qquad \text{and} \qquad \mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\theta(\mathbf{r},t) \; , \qquad (3)$$

inserting these formulas into Eq. (2) one finds

$$\frac{\partial}{\partial t} n + \boldsymbol{\nabla}\cdot(n\,\mathbf{v}) = 0 \; , \qquad (4)$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \boldsymbol{\nabla}\left[\frac{1}{2} m v^2 + U(\mathbf{r}) + gn - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} \right] = 0 \; . \qquad (5)$$

Madelung transformation (II)

Eqs. (4) and (5) are, respectively, the equation of continuity and the equation of conservation of linear momentum for a irrotational and inviscid fluid.
The zero-temperature equation of state of this superfluid, i.e. the local chemical potential as a function of the local density and its derivatives, can be written as

$$\mu(n, \nabla^2 n) = gn - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \ , \tag{6}$$

where the second term, which is usually called quantum pressure, becomes negligible in the high-density regime.
Notice that the local velocity field

$$\mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m} \boldsymbol{\nabla} \theta(\mathbf{r}, t) \tag{7}$$

is by definition irrotational, i.e. such that

$$\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0} \ . \tag{8}$$

Superfluid hydrodynamics (I)

Quantum effects are encoded not only in the equation of state

$$\mu(n, \nabla^2 n) = gn - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \ , \tag{9}$$

also into the properties of the local field $\mathbf{v}(\mathbf{r}, t)$: it is proportional to the gradient of a scalar field, $\theta(\mathbf{r}, t)$, that is the angle of the phase of the single-valued complex wavefunction $\psi_0(\mathbf{r}, t)$. Indeed, one gets

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r} = 2\pi \frac{\hbar}{m} k \tag{10}$$

for any closed contour $\mathcal{C}$, with k an integer number. In other words, the circulation is quantized in units of $\hbar/m$, and this property is strictly related to the existence of quantized vortices.

Superfluid hydrodynamics (II)

Let us assume that $U(\mathbf{r}) = 0$. The equations of superfluid hydrodynamics become

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\,\mathbf{v}) = 0 \,, \qquad (11)$$

$$m \frac{\partial}{\partial t}\mathbf{v} + \nabla \left[\frac{1}{2} m v^2 + g n - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right] = 0 \,. \qquad (12)$$

We now set

$$n(\mathbf{r}, t) = n_{eq} + \delta n(\mathbf{r}, t) \,, \qquad (13)$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{0} + \delta \mathbf{v}(\mathbf{r}, t) \,, \qquad (14)$$

where $\delta n(\mathbf{r}, t)$ and $\delta \mathbf{v}(\mathbf{r}, t)$ represent small variations with respect to the uniform and constant stationary configuration n_{eq}.

Superfluid hydrodynamics (III)

In this way, neglecting quadratic terms in the variations (linearization) from Eqs. (11) and (12) we get the linear equations of motion

$$\begin{aligned} \frac{\partial}{\partial t}\delta n + n_{eq}\boldsymbol{\nabla}\cdot\delta\mathbf{v} &= 0\ , & (15) \\ \frac{\partial}{\partial t}\delta\mathbf{v} + \frac{c_s^2}{n_{eq}}\boldsymbol{\nabla}\delta n - \frac{\hbar^2}{4m^2 n_{eq}}\boldsymbol{\nabla}(\nabla^2\delta n) &= \mathbf{0}\ , & (16) \end{aligned}$$

where c_s is the sound velocity of the bosonic superfluid, given by

$$m c_s^2 = g\, n_{eq}\ . \qquad (17)$$

The linear equations of motion can be arranged in the form of the following wave equation

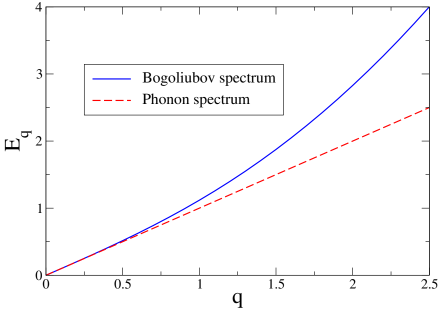
$$\left[\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 + \frac{\hbar^2}{4m^2}\nabla^4\right]\delta n(\mathbf{r},t) = 0\ . \qquad (18)$$

Superfluid hydrodynamics (IV)

The wave equation admits monochromatic plane-wave solutions, where the frequency ω and the wave vector $\mathbf{q}$ are related by the dispersion formula $\omega = \omega(q)$ given by

$$E_q = \hbar\,\omega(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)} \,. \tag{19}$$

This is called Bogoliubov spectrum of elementary excitations.



In the above Figure there is the Bogoliubov spectrum and its low-momenta ($q \ll 1$) phonon spectrum $E_q = c_s\,\hbar q$.