Lesson 7 - Superfluids
Unit 7.1 Time-dependent Gross-Pitaevskii equation

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Time-dependent Hartree equation (I)

The time-dependent Schrödinger equation of a many-body system with Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \nabla_i^2 + U(\mathbf{r}_i) \right) + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} V(\mathbf{r}_i - \mathbf{r}_j) \ , \tag{1}$$

where $U(\mathbf{r})$ is the external potential and $V(\mathbf{r} - \mathbf{r}')$ is the inter-atomic potential, is given by

$$i\hbar \frac{\partial}{\partial t} \mathbf{\Psi}(\mathbf{r}_1, ..., \mathbf{r}_N, t) = \hat{H} \mathbf{\Psi}(\mathbf{r}_1, ..., \mathbf{r}_N, t), \tag{2}$$

where $\mathbf{\Psi}(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ is the time-dependent many-body wavefuction. This time-dependent many-body Schrödinger equation is the Euler-Lagrange equation of the following many-body action functional

$$S = \int dt \ d^3\mathbf{r}_1 \ ... \ d^3\mathbf{r}_N \ \mathbf{\Psi}^*(\mathbf{r}_1, ..., \mathbf{r}_N, t) \left(i\hbar \frac{\partial}{\partial t} - \hat{H} \right) \mathbf{\Psi}(\mathbf{r}_1, ..., \mathbf{r}_N, t) \ . \tag{3}$$

Time-dependent Hartree equation (II)

In the case of a pure Bose-Einstein condensate one assumes all bosons in the same time-dependent single-particle orbital (Hartree approximation)

$$\mathbf{\Psi}(\mathbf{r}_1, ..., \mathbf{r}_N, t) = \prod_{i=1}^{N} \psi(\mathbf{r}_i, t) \ . \tag{4}$$

Inserting this ansatz into the many-body action functional one gets

$$\begin{aligned} S &= N \int dt \ d^3\mathbf{r} \ \psi^*(\mathbf{r}, t) \Big(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - U(\mathbf{r}) \\ &- \frac{N-1}{2} \int d^3\mathbf{r}' \ |\psi(\mathbf{r}', t)|^2 V(\mathbf{r} - \mathbf{r}') \Big) \psi(\mathbf{r}, t) \ . \end{aligned} \tag{5}$$

The Euler-Lagrange equation of the previous action functional reads

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) + (N-1) \int d^3\mathbf{r}' \ |\psi(\mathbf{r}', t)|^2 V(\mathbf{r} - \mathbf{r}') \right] \psi(\mathbf{r}, t) \ . \tag{6}$$

This is the time-dependent Hartree equation for N identical bosons in the same single-particle state $\psi(\mathbf{r}, t)$.

Time-dependent Gross-Pitaevskii equation (I)

In the case of dilute gases we assume (Fermi pseudo-potential) that

$$V(\mathbf{r}) \simeq g\ \delta^{(3)}(\mathbf{r}) \tag{7}$$

with $\delta^{(3)}(\mathbf{r})$ the Dirac delta function and, by construction,

$$g = \int V(\mathbf{r})\ d^3\mathbf{r}\ . \tag{8}$$

In this way, from the Hatree action functional we obtain

$$S = N \int dt\ d^3\mathbf{r}\ \psi^*(\mathbf{r}, t)\Big(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - U(\mathbf{r}) - \frac{N-1}{2} g|\psi(\mathbf{r}, t)|^2 \Big)\psi(\mathbf{r}, t)\ , \tag{9}$$

that is the Gross-Pitaevskii action functional, whose Euler-Lagrange equation reads

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1) g|\psi(\mathbf{r}, t)|^2 \right] \psi(\mathbf{r}, t)\ . \tag{10}$$

This is the time-dependent Gross-Pitaevskii equation, derived for the fist time by Eugene Gross and Lev Pitaevskii in 1961.