Lesson 3 - Matter-Radiation Interaction
Unit 3.3 Absorption and stimulated emission

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Absorption (I)

We now consider the excitation from the atomic state $|a\rangle$ to the atomic state $|b\rangle$ due to the absorption of one photon. Thus we suppose that the initial state is

$$|I\rangle = |a\rangle|n_{\mathbf{k}s}\rangle \ , \tag{1}$$

while the final state is

$$|F\rangle = |b\rangle|n_{\mathbf{k}s} - 1\rangle \ , \tag{2}$$

where $E_a < E_b$. From the Golden rule one finds

$$W^{absorp}_{ab,\mathbf{k}s} = \frac{2\pi}{\hbar} \left(\frac{e}{m}\right)^2 \left(\frac{\hbar}{2\epsilon_0 \omega_k V}\right) n_{\mathbf{k}s} \, |\varepsilon_{\mathbf{k}s} \cdot \langle b|\hat{\mathbf{p}}|a\rangle|^2 \, \delta(E_a + \hbar\omega_k - E_b) \ , \tag{3}$$

because

$$\langle F|\hat{\mathbf{p}} \, \hat{a}_{\mathbf{k}'s'}|I\rangle = \sqrt{n_{\mathbf{k}s}} \, \langle b|\hat{\mathbf{p}}|a\rangle \ \delta_{\mathbf{k}',\mathbf{k}} \ \delta_{s',s} \ , \qquad \langle F|\hat{\mathbf{p}} \, \hat{a}^+_{\mathbf{k}'s'}|I\rangle = 0 \ . \tag{4}$$

Absorption (II)

We can follow the procedure of the previous Unit to get

$$W_{ab,\mathbf{k}s}^{absorp} = \frac{\pi\omega_{ba}^2}{V\epsilon_0\omega_k}\, n_{\mathbf{k}s}\, |\varepsilon_{\mathbf{k}s}\cdot\langle b|e\,\mathbf{r}|a\rangle|^2\,\delta(\hbar\omega_{ba}-\hbar\omega_k)\ . \qquad (5)$$

Again the delta function can be eliminated by integrating over the final photon states but here one must choose the functional dependence of $n_{\mathbf{k}s}$. We simply set

$$n_{\mathbf{k}s} = n(\omega_k)\ , \qquad (6)$$

and after integration over $\mathbf{k}$ and s, from Eq. (5) we get

$$W_{ab}^{absorp} = \frac{\omega_{ba}^3}{3\pi\epsilon_0\hbar c^3}\,|\langle b|\mathbf{d}|a\rangle|^2\, n(\omega_{ba}) = W_{ba}^{spont}\, n(\omega_{ba})\ . \qquad (7)$$

Absorption (III)

For a thermal distribution of photons, with $\rho(\omega)$ the energy density per unit of angular frequency specified by the thermal-equilibrium Planck formula

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\, n(\omega)\ , \qquad n(\omega) = \frac{1}{e^{\hbar\omega/(k_B T)} - 1}\ , \tag{8}$$

where k_B is the Boltzmann constant and T the absolute temperature, we can also write

$$W_{ab}^{absorp} = |\langle b|\mathbf{d}|a\rangle|^2\, \frac{1}{3\epsilon_0 \hbar^2}\, \rho(\omega_{ba}) = W_{ba}^{spont}\, \frac{\pi^2 c^3}{\hbar\omega_{ba}^3} \rho(\omega_{ba})\ . \tag{9}$$

Stimulated emission (I)

Finally, we consider the stimulated emission of a photon from the atomic state $|b\rangle$ to the atomic state $|a\rangle$. Thus we suppose that the initial state is

$$|I\rangle = |b\rangle|n_{\mathbf{k}s}\rangle \ , \tag{10}$$

while the final state is

$$|F\rangle = |a\rangle|n_{\mathbf{k}s} + 1\rangle \ , \tag{11}$$

where $E_b > E_a$. From the Golden rule one finds

$$W^{stimul}_{ba,\mathbf{k}s} = \frac{2\pi}{\hbar} \left(\frac{e}{m}\right)^2 \left(\frac{\hbar}{2\epsilon_0 \omega_k V}\right) (n_{\mathbf{k}s} + 1) \ |\varepsilon_{\mathbf{k}s} \cdot \langle a|\hat{\mathbf{p}}|b\rangle|^2 \, \delta(E_b - E_a - \hbar\omega_k) \ . \tag{12}$$

Note that with respect to Eq. (3) in Eq. (12) there is the factor $n_{\mathbf{k}s} + 1$ instead of $n_{\mathbf{k}s}$.

Stimulated emission (II)

It is straightforward to follow the previous procedure obtaing

$$W_{ba}^{stimul} = \frac{\omega_{ba}^3}{3\pi\epsilon_0\hbar c^3} \, |\langle a|\mathbf{d}|b\rangle|^2 \, (n(\omega_{ba}) + 1) = W_{ab}^{absorp} + W_{ba}^{spont} \ , \quad (13)$$

which shows that the probability W_{ba}^{stimul} of stimulated emission reduces to the spontaneous one W_{ba}^{spont} when $n(\omega_{ba}) = 0$. It is then useful to introduce

$$\tilde{W}_{ba}^{stimul} = W_{ba}^{stimul} - W_{ba}^{spont} \quad (14)$$

which is the effective stimulated emission, i.e. the stimulated emission without the contribution due to the spontaneous emission. Clearly, for a very large number of photons ($n(\omega_{ba}) \gg 1$) one gets $\tilde{W}_{ba}^{stimul} \simeq W_{ba}^{stimul}$. Moreover, for a thermal distribution of photons, with the energy density per unit of angular frequency $\rho(\omega)$, we can also write

$$\tilde{W}_{ba}^{stimul} = W_{ab}^{absorp} = W_{ba}^{spont} \frac{\pi^2 c^3}{\hbar\omega_{ba}^3} \rho(\omega_{ba}) \ . \quad (15)$$

Stimulated emission (III)

For a thermal distribution of photons, with the energy density per unit of angular frequency $\rho(\omega)$, we can also write

$$\tilde{W}_{ba}^{stimul} = W_{ab}^{absorp} = W_{ba}^{spont} \frac{\pi^2 c^3}{\hbar \omega_{ba}^3} \rho(\omega_{ba}) \ . \tag{16}$$

Remarkably, the probability of stimulated emission is different from zero only if the emitted photon is in the same single-particle state $|\mathbf{k}s\rangle$ of the stimulating ones, apart when the stimulating light is the vacuum $|0\rangle$.
In the stimulated emission the emitted photon is said to be "coherent" with the stimulating ones, having the same frequency and the same direction.