Appendix D

Fermi Golden Rule

Let us consider a quantum system described by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_I \ , \tag{D.1}$$

where $\hat{H}_0$ is the Hamiltonian of the unperturbed part while $\hat{H}_I$ is the Hamiltonian of the perturbation.

A generic eigenstate $|\phi_n\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ satisfies the stationary Schrödinger equation

$$\hat{H}_0|\phi_n\rangle = E_n|\phi_n\rangle \ , \tag{D.2}$$

where E_n is the corresponding eigenstate of the unperturbed Hamiltonian $\hat{H}_0$.

If the perturbation is zero, i.e. if $\hat{H}_I = 0$, then the time evolution of $|\phi_n\rangle$ is simple:

$$|\phi_n(t)\rangle = e^{-iE_nt/\hbar}|\phi_n(0)\rangle \ . \tag{D.3}$$

Clearly, in this case, it is zero the probability of finding the eigenstate $|\phi_n\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ in another eigenstate $|\phi_l\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$.

If instead the perturbation is not zero, i.e. if $\hat{H}_I \neq 0$, then the time evolution of $|\phi_n\rangle$ is, in general, quit complicated because, usually, $|\phi_n\rangle$ is not an eigenstate of the total Hamiltonian $\hat{H}$. The Fermi golden rule is relevant in this case because it gives a way to calculate the probability of finding the eigenstate $|\phi_n\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ into another eigenstate $|\phi_l\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$.

A generic time-dependent state $|\psi(t)\rangle$ of the total Hamiltonian $\hat{H}$ of Eq. (D.1) satisfies the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left(\hat{H}_0 + \hat{H}_I\right)|\psi(t)\rangle \ . \tag{D.4}$$

This state $|\psi(t)\rangle$ can be expanded in the orthonormal basis of the time-independent eigenstates $|\phi_j(0)\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ as follows

$$|\psi(t)\rangle = \sum_j c_j(t)\ e^{-iE_jt/\hbar}\ |\phi_j(0)\rangle \ , \tag{D.5}$$

where the complex coefficients $c_j(t)$ are all equal to one only in the very special case of $\hat{H}_I = 0$. For the sake of simplicity we approximate Eq. (D.5) adopting the two-mode approximation which involves only two eigenstates $|\phi_I(0)\rangle$ and $|\phi_F(0)\rangle$ of the unperturbed Hamiltonian H_0:

$$|\psi(t)\rangle = \sum_{j=I,F} c_j(t)\ e^{-iE_j t/\hbar}\ |\phi_j(0)\rangle\ , \tag{D.6}$$

assuming that at $t = 0$ the state $|\psi(0)\rangle$ of the system is in the initial state $|\phi_I(0)\rangle$, namely $c_I(0) = 1$ and $c_F(0) = 0$. Here $|\phi_F(0)\rangle$ is our final state, and clearly $\langle\phi_I(0)|\phi_F(0)\rangle = 0$.

Inserting the expression (D.6) into Eq. (D.4) and the bra $\langle\phi_F(0)|$ on the left side of the resulting formula we obtain

$$i\hbar\ \dot{c}_F(t) = \langle\phi_F(0)|\hat{H}_I|\phi_I(0)\rangle\ e^{i\omega_{IF}t}\ , \tag{D.7}$$

where $\omega_{IF} = (E_I - E_F)/\hbar$. The solution of this equation is given by

$$c_F(t) = \frac{\langle F|\hat{H}_I|I(0)\rangle}{i\hbar} \int_0^t e^{i\omega_{IF}t'}\ dt' = \frac{\langle F|\hat{H}_I|I\rangle}{\hbar\omega_{IF}} \left(1 - e^{i\omega_{IF}t}\right)\ , \tag{D.8}$$

where we set $|I\rangle = |\phi_I(0)\rangle$ and $F = |\phi_F(0)\rangle$. It follows that

$$|c_F(t)|^2 = \frac{|\langle\phi_F(0)|\hat{H}_I|\phi_F(0)\rangle|^2}{\hbar^2\omega_{IF}^2} 4\sin^2\left(\omega_{IF}t/2\right)\ . \tag{D.9}$$

We can now introduce the transition probability per unit time

$$\frac{|c_F(t)|^2}{t} = \frac{1}{\hbar^2}\ t\ \frac{\sin^2(\omega_{IF}t/2)}{(\omega_{IF}t/2)^2}\ . \tag{D.10}$$

Because the Dirac delta function $\delta(x)$ can be written as

$$\delta(x) = \frac{1}{2\pi} \lim_{t\to+\infty} t\ \frac{\sin^2(x\ t)}{(x\ t)^2}\ , \tag{D.11}$$

the asymptotic transition probability per unit time

$$W_{IF} = \lim_{t\to+\infty} \frac{|c_F(t)|^2}{t} \tag{D.12}$$

reads

$$W_{IF} = \frac{2\pi}{\hbar}|\langle F|H_I|I\rangle|^2\ \delta(E_I - E_F)\ , \tag{D.13}$$

which is the Fermi golden rule.