# Lesson 3 - Matter-Radiation Interaction
# Unit 3.2 Fermi golden rule and spontaneous emission

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# Fermi golden rule (I)

We have seen that the total Hamiltonian of the matter-radiation system in the dipole approximation is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_D \ , \tag{1}$$

where

$$\hat{H}_0 = \hat{H}_{matt} + \hat{H}_{rad} \tag{2}$$

is the unperturbed Hamiltonian and $\hat{H}_D$ is the dipole Hamiltonian, which couples matter and radiation.
**Fermi golden rule**: Given the initial $|I\rangle$ and final $|F\rangle$ eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ under the presence to the perturbing Hamiltonian $\hat{H}_D$, the probability per unit time of the transition from $|I\rangle$ to $|F\rangle$ is given by

$$W_{IF} = \frac{2\pi}{\hbar} \, |\langle F|\hat{H}_D|I\rangle|^2 \, \delta(E_I - E_F) \ , \tag{3}$$

with the constraint of energy conservation.

# Spontaneous emission (I)

Let us now apply the Fermi golden rule to the very interesting case of the hydrogen atom in the state $|b\rangle$ and the radiation field in the vacuum state $|0\rangle$. We are thus supposing that the initial state is

$$|I\rangle = |b\rangle|0\rangle \ . \tag{4}$$

Notice that, because we are considering the hydrogen atom, one has

$$\hat{H}_{matt}|b\rangle = E_b|b\rangle \ , \tag{5}$$

where

$$E_b = -\frac{13.6 \text{ eV}}{n_b^2} \tag{6}$$

is the well-known quantization formula of the nonrelativistic hydrogen atom with quantum number $n_b = 1, 2, 3, ....$ In addition we suppose that the final state is

$$|F\rangle = |a\rangle|\mathbf{k}s\rangle \ , \tag{7}$$

i.e. the final atomic state is $|a\rangle$ and the final photon state is $|\mathbf{k}s\rangle = |1_{\mathbf{k}s}\rangle = \hat{a}^{+}_{\mathbf{k}s}|0\rangle$.

# Spontaneous emission (II)

From Eq. (3) one finds

$$W^{spont}_{ba,\mathbf{k}s} = \frac{2\pi}{\hbar}\left(\frac{e}{m}\right)^2\left(\frac{\hbar}{2\epsilon_0\omega_k V}\right)|\varepsilon_{\mathbf{k}s}\cdot\langle a|\hat{\mathbf{p}}|b\rangle|^2\,\delta(E_b - E_a - \hbar\omega_k)\ , \quad (8)$$

because

$$\hat{a}_{\mathbf{k}'s'}|I\rangle = \hat{a}_{\mathbf{k}'s'}|b\rangle|0\rangle = |b\rangle\hat{a}_{\mathbf{k}'s'}|0\rangle = 0\ , \quad (9)$$

while

$$\hat{a}^{+}_{\mathbf{k}'s'}|I\rangle = \hat{a}^{+}_{\mathbf{k}'s'}|b\rangle|0\rangle = |b\rangle\hat{a}^{+}_{\mathbf{k}'s'}|0\rangle = |b\rangle|\mathbf{k}'s'\rangle\ , \quad (10)$$

and consequently

$$\langle F|\hat{\mathbf{p}}\,\hat{a}_{\mathbf{k}'s'}|I\rangle = 0\ , \qquad \langle F|\hat{\mathbf{p}}\,\hat{a}^{+}_{\mathbf{k}'s'}|I\rangle = \langle a|\hat{\mathbf{p}}|b\rangle\ \delta_{\mathbf{k}',\mathbf{k}}\ \delta_{s',s}\ . \quad (11)$$

# Spontaneous emission (III)

On the basis of Heisenberg equation of motion of the linear momentum operator $\hat{\mathbf{p}}$ of the electron

$$\frac{\hat{\mathbf{p}}}{m} = \frac{d\mathbf{r}}{dt} = \frac{1}{i\hbar}[\mathbf{r}, \hat{H}_{matt}] \ , \tag{12}$$

we get

$$\begin{aligned} \langle a|\hat{\mathbf{p}}|b\rangle &= \langle a|m\frac{1}{i\hbar}[\mathbf{r}, \hat{H}_{matt}]|b\rangle = \frac{m}{i\hbar}\langle a|\mathbf{r}\hat{H}_{matt} - \hat{H}_{matt}\mathbf{r}|b\rangle \\ &= \frac{m}{i\hbar}(E_b - E_a)\langle b|\mathbf{r}|a\rangle = -im\omega_{ba}\,\langle a|\mathbf{r}|b\rangle \ , \end{aligned} \tag{13}$$

where $\omega_{ba} = (E_b - E_a)/\hbar$, and consequently

$$W^{spont}_{ba,\mathbf{ks}} = \frac{\pi\omega_{ba}^2}{V\epsilon_0\omega_k}\,|\varepsilon_{\mathbf{ks}} \cdot \langle a|e\,\mathbf{r}|b\rangle|^2\,\delta(\hbar\omega_{ba} - \hbar\omega_k) \ . \tag{14}$$

# Spontaneous emission (IV)

The delta function is eliminated by integrating over the final photon states

$$W_{ba}^{spont} = \sum_{\mathbf{k}} \sum_{s} W_{ba,\mathbf{k}s}^{spont} = V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{s=1,2} W_{ba,\mathbf{k}s}^{spont}$$
$$= \frac{V}{8\pi^3} \int dk k^2 \int d\Omega \sum_{s=1,2} W_{ba,\mathbf{k}s}^{spont} \,, \qquad (15)$$

where $d\Omega$ is the differential solid angle.
Because $\varepsilon_{\mathbf{k}1}$, $\varepsilon_{\mathbf{k}2}$ and $\mathbf{n} = \mathbf{k}/k$ form a orthonormal system of vectors, setting $\mathbf{r}_{ab} = \langle a|\mathbf{r}|b\rangle$ one finds

$$|\mathbf{r}_{ab}|^2 = |\varepsilon_{\mathbf{k}1} \cdot \mathbf{r}_{ab}|^2 + |\varepsilon_{\mathbf{k}2} \cdot \mathbf{r}_{ab}|^2 + |\mathbf{n} \cdot \mathbf{r}_{ab}|^2 = \sum_{s=1,2} |\varepsilon_{\mathbf{k}s} \cdot \mathbf{r}_{ab}|^2 + |\mathbf{r}_{ab}|^2 \cos^2(\theta) \,, \qquad (16)$$

where $\theta$ is the angle between $\mathbf{r}_{ba}$ and $\mathbf{n}$.

# Spontaneous emission (V)

It follows immediately

$$\sum_{s=1,2} |\varepsilon_{\mathbf{k}s} \cdot \mathbf{r}_{ab}|^2 = |\mathbf{r}_{ab}|^2(1 - \cos^2(\theta)) = |\mathbf{r}_{ab}|^2 \sin^2(\theta) = |\langle a|\mathbf{r}|b\rangle|^2 \sin^2(\theta) . \tag{17}$$

In addition, in spherical coordinates one can choose $d\Omega = \sin(\theta) d\theta d\phi$, with $\theta \in [0, \pi]$ the zenith angle of colatitude and $\phi \in [0, 2\pi]$ the azimuth angle of longitude, and then

$$\int d\Omega \sin^2(\theta) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3(\theta) = \frac{8\pi}{3} . \tag{18}$$

In this way from Eq. (15) we finally obtain

$$W_{ba}^{spont} = \frac{\omega_{ba}^3}{3\pi\epsilon_0 \hbar c^3} |\langle a|\mathbf{d}|b\rangle|^2 , \tag{19}$$

where the $\mathbf{d} = -e\,\mathbf{r}$ is the classical electric dipole momentum of the hydrogen atom, i.e. the dipole of the electron-proton system where $\mathbf{r}$ is the position of the electron of charge $-e < 0$ with respect to the proton of charge $e > 0$, and $\langle a|\mathbf{d}|b\rangle = -\langle a|e\,\mathbf{r}|b\rangle$ is the so-called dipole transition element.