Lesson 3 - Matter-Radiation Interaction
Unit 3.1 Minimal coupling and dipole approximation

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Structure of Matter - MSc in Physics

Minimal coupling (I)

Let us consider the hydrogen atom with Hamiltonian

$$\hat{H}_{matt} = \frac{\hat{\mathbf{p}}^2}{2m} - \frac{e^2}{4\pi\epsilon_0|\mathbf{r}|} \ , \tag{1}$$

where $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$ is the linear momentum operator of the electron in the state $|\mathbf{p}\rangle$, and $e > 0$ is the modulus of the electric charge of the electron. The minimal coupling with the electromagnetic field is obtained with the substitution

$$\hat{\mathbf{p}} \to \hat{\mathbf{p}} + e\hat{\mathbf{A}}(\mathbf{r}, t) \tag{2}$$

where $\mathbf{A}(\mathbf{r}, t)$ is the vector potential of the electromagnetic field.
In this way we have

$$\begin{aligned} \hat{H}_{matt,shift} &= \frac{\left(\hat{\mathbf{p}} + e\hat{\mathbf{A}}(\mathbf{r}, t)\right)^2}{2m} - \frac{e^2}{4\pi\epsilon_0|\mathbf{r}|} \\ &= \hat{H}_{matt} + \frac{e}{m}\hat{\mathbf{A}}(\mathbf{r}, t) \cdot \hat{\mathbf{p}} + \frac{e^2}{2m}\hat{\mathbf{A}}(\mathbf{r}, t)^2 \ . \end{aligned} \tag{3}$$

Dipole approximation (I)

The dipole approximation means

$$\hat{H}_{matt,shift} \simeq \hat{H}_{matt} + \hat{H}_D \ , \tag{4}$$

where

$$H_D = \frac{e}{m}\hat{\mathbf{A}}(\mathbf{0},0)\cdot\hat{\mathbf{p}} \ . \tag{5}$$

This means that one neglects the term $(e^2/2m)\hat{\mathbf{A}}(\mathbf{r},t)^2$ because it is quadratic correction with respect to the weak vector potential and one uses $\hat{\mathbf{A}}(\mathbf{0},0)$ instead of $\hat{\mathbf{A}}(\mathbf{r},t)$.
The latter assumption, which corresponds to

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 1 + i\mathbf{k}\cdot\mathbf{r} + \frac{1}{2}(i\mathbf{k}\cdot\mathbf{r})^2 + \ ... \ \simeq 1 \ , \tag{6}$$

is reliable if $\mathbf{k}\cdot\mathbf{r} \ll 1$, namely if the electromagnetic radiation has a wavelength $\lambda = 2\pi/|\mathbf{k}|$ very large compared to the linear dimension R of the atom. Indeed, the approximation is fully justified in atomic physics where $\lambda \simeq 10^{-7}$ m and $R \simeq 10^{-10}$ m.

Quantum electrodynamics (I)

The total Hamiltonian of the matter-radiation system in the dipole approximation is then given by

$$\hat{H} = \hat{H}_0 + \hat{H}_D \ , \tag{7}$$

where

$$\hat{H}_0 = \hat{H}_{matt} + \hat{H}_{rad} \tag{8}$$

is the unperturbed Hamiltonian, such that

$$\hat{H}_{matt} = \frac{\hat{\mathbf{p}}^2}{2m} - \frac{e^2}{4\pi\epsilon_0|\mathbf{r}|} \ , \tag{9}$$

is the matter Hamiltonian, while the radiation Hamiltonian reads

$$\hat{H}_{rad} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \, \hat{a}^+_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} \ , \tag{10}$$

where $\hat{a}_{\mathbf{k}s}$ and $\hat{a}_{\mathbf{k}s}$ are the annihilation and creation operators of the photon in the state $|\mathbf{k}s\rangle$.

Quantum electrodynamics (II)

The eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ are of the form

$$|a\rangle|\dots n_{\mathbf{k}s}\dots\rangle = |a\rangle \otimes |\dots n_{\mathbf{k}s}\dots\rangle \tag{11}$$

where $|a\rangle$ is the eigenstate of $\hat{H}_{matt}$ with eigenvalue E_a and $|\dots n_{\mathbf{k}s}\dots\rangle$ it the eigenstate of $\hat{H}_{rad}$ with eigenvalue $\sum_{\mathbf{k}s}\hbar\omega_k n_{\mathbf{k}s}$, i.e.

$$\begin{aligned}
\hat{H}_0|a\rangle|\dots n_{\mathbf{k}r}\dots\rangle &= \left(\hat{H}_{matt}+\hat{H}_{rad}\right)|a\rangle|\dots n_{\mathbf{k}s}\dots\rangle \\
&= \left(E_a+\sum_{\mathbf{k}s}\hbar\omega_k n_{\mathbf{k}s}\right)|a\rangle|\dots n_{\mathbf{k}s}\dots\rangle\,. \qquad (12)
\end{aligned}$$