Lesson 2 - Quantum Properties of Light
Unit 2.3 Casimir effect

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Zero-point energy of the electromagnetic field (I)

The zero-point energy of the electromagnetic field in a region of volume V can be written as

$$E_{vac} = \sum_{\mathbf{k}} \sum_{s} \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} = V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} \quad (1)$$

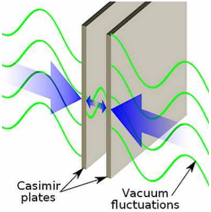
by using the dispersion relation $\omega_k = ck = c\sqrt{k_x^2 + k_y^2 + k_z^2}$.
Considering a region with the shape of a parallelepiped of length L along both x and y and length a along z, the volume V is given by $V = L^2 a$ and the vacuum energy E_{vac} in the region is

$$\begin{aligned} E_{vac} &= \hbar c \int_{-\infty}^{+\infty} \frac{L\, dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{L\, dk_y}{2\pi} \int_{-\infty}^{+\infty} \frac{a\, dk_z}{2\pi} \sqrt{k_x^2 + k_y^2 + k_z^2} \\ &= \frac{\hbar c}{2\pi} L^2 \int_0^{\infty} dk_\parallel k_\parallel \left[\int_0^{\infty} dn \sqrt{k_\parallel^2 + \frac{n^2\pi^2}{a^2}} \right], \qquad (2) \end{aligned}$$

where the second expression is obtained setting $k_\parallel = \sqrt{k_x^2 + k_y^2}$ and $n = (a/\pi) k_z$.

Zero-point energy between Casimir plates (I)

Let us now consider the presence of two perfect metallic plates with the shape of a square of length L having parallel faces lying in the (x, y) plane at distance a. There are usally called Casimir plates.



Along the z axis the stationary standing waves of the electromagnetic field vanishes on the metal plates and the k_z component of the wavevector $\mathbf{k}$ is no more a continuum variabile but it is quantized via

$$k_z = n\frac{\pi}{a} \ , \tag{3}$$

where now $n = 0, 1, 2, ...$ is an integer number, and not a real number as in Eq. (2).

Zero-point energy between Casimir plates (II)

In the case of Casimir plates the zero-point energy in the volume $V = L^2 a$ between the two plates reads

$$E'_{vac} = \frac{\hbar c}{2\pi} L^2 \int_0^\infty dk_\parallel \, k_\parallel \left[\frac{k_\parallel}{2} + \sum_{n=1}^{\infty} \sqrt{k_\parallel^2 + \frac{n^2\pi^2}{a^2}} \right] . \tag{4}$$

The difference between E'_{vac} and E_{vac} divided by L^2 gives the net energy per unit surface area $\mathcal{E}$, namely

$$\mathcal{E} = \frac{E'_{vac} - E_{vac}}{L^2} = \frac{\hbar c}{2\pi} \left(\frac{\pi}{a}\right)^3 \left[\frac{1}{2} A(0) + \sum_{n=1}^{\infty} A(n) - \int_0^\infty dn \, A(n) \right] , \tag{5}$$

where we have defined

$$A(n) = \int_0^{+\infty} d\zeta \, \zeta \, \sqrt{\zeta^2 + n^2} = \frac{1}{3} \left[(n^2 + \infty)^{3/2} - n^2 \right] , \tag{6}$$

with $\zeta = (a/\pi) k_\parallel$.

The Casimir force (I)

After some analytical manipulations one finds an explicit formula for the energy difference $\mathcal{E} = \frac{E'_{vac} - E_{vac}}{L^2}$ per unit area.
From this energy difference $\mathcal{E}$ one deduces that there is an attractive force per unit area $\mathcal{F}$ between the two plates, given by

$$\mathcal{F} = -\frac{d\mathcal{E}}{da} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} \ . \tag{7}$$

Numerically this result, predicted in 1948 by Hendrik Casimir during his research activity at the Philips Physics Laboratory in Eindhoven, is very small

$$\mathcal{F} = -\frac{1.30 \cdot 10^{-27}\mathrm{N\ m}^2}{a^4} \ . \tag{8}$$

Nevertheless, it has been experimentally verified by Steven Lamoreaux in 1997 at the University of Washington and by Giacomo Bressi, Gianni Carugno, Roberto Onofrio, and Giuseppe Ruoso in 2002 at the University of Padova.