Lesson 2 - Quantum Properties of Light
Unit 2.2 Gas of photons and Planck law

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Partition function of photons (I)

Let us consider the electromagnetic field in thermal equilibrium with a bath at the temperature T. The relevant quantity to calculate all thermodynamical properties of the system is the grand-canonical partition function $\mathcal{Z}$, given by

$$\mathcal{Z} = Tr[e^{-\beta(\hat{H}-\mu\hat{N})}] \tag{1}$$

where $\beta = 1/(k_B T)$ with $k_B = 1.38 \cdot 10^{-23}$ J/K the Boltzmann constant,

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \, \hat{N}_{\mathbf{k}s} \ , \tag{2}$$

is the quantum Hamiltonian without the zero-point energy,

$$\hat{N} = \sum_{\mathbf{k}} \sum_{s} \hat{N}_{\mathbf{k}s} \tag{3}$$

is the total number operator, and μ is the chemical potential, fixed by the conservation of the particle number.

Partition functions of photons (II)

For photons $\mu = 0$ and consequently the number of photons is not fixed. This implies that

$$
\begin{aligned}
\mathcal{Z} &= \sum_{\{n_{\mathbf{k}s}\}} \langle \dots n_{\mathbf{k}s} \dots | e^{-\beta \hat{H}} | \dots n_{\mathbf{k}s} \dots \rangle \\
&= \sum_{\{n_{\mathbf{k}s}\}} \langle \dots n_{\mathbf{k}s} \dots | e^{-\beta \sum_{\mathbf{k}s} \hbar\omega_k \hat{N}_{\mathbf{k}s}} | \dots n_{\mathbf{k}s} \dots \rangle \\
&= \sum_{\{n_{\mathbf{k}s}\}} e^{-\beta \sum_{\mathbf{k}s} \hbar\omega_k n_{\mathbf{k}s}} = \sum_{\{n_{\mathbf{k}s}\}} \prod_{\mathbf{k}s} e^{-\beta\hbar\omega_k n_{\mathbf{k}s}} \\
&= \prod_{\mathbf{k}s} \sum_{n_{\mathbf{k}s}} e^{-\beta\hbar\omega_k n_{\mathbf{k}s}} = \prod_{\mathbf{k}s} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega_k n} \\
&= \prod_{\mathbf{k}s} \frac{1}{1 - e^{-\beta\hbar\omega_k}} .
\end{aligned}
\tag{4}
$$

Thermal energy of photons (I)

Quantum statistical mechanics dictates that the thermal average of any operator $\hat{A}$ is obtained as

$$\langle \hat{A} \rangle_T = \frac{1}{\mathcal{Z}} Tr[\hat{A}\, e^{-\beta(\hat{H}-\mu\hat{N})}] \, . \tag{5}$$

In our case the calculations are simplified because $\mu = 0$. Let us suppose that $\hat{A} = \hat{H}$, it is then quite easy to show that

$$\langle \hat{H} \rangle_T = \frac{1}{\mathcal{Z}} Tr[\hat{H}\, e^{-\beta\hat{H}}] = -\frac{\partial}{\partial\beta} \ln\left(Tr[e^{-\beta\hat{H}}]\right) = -\frac{\partial}{\partial\beta} \ln(\mathcal{Z}) \, . \tag{6}$$

By using Eq. (4) we immediately obtain

$$\ln(\mathcal{Z}) = -\sum_{\mathbf{k}} \sum_{s} \ln\left(1 - e^{-\beta\hbar\omega_k}\right) \, , \tag{7}$$

and finally from Eq. (6) we get

$$\langle \hat{H} \rangle_T = \sum_{\mathbf{k}} \sum_{s} \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \, \langle \hat{N}_{\mathbf{k}s} \rangle_T \, . \tag{8}$$

Thermal energy of photons (II)

In the continuum limit, where

$$\sum_{\mathbf{k}} \to V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ , \tag{9}$$

with V the volume, and taking into account that $\omega_k = ck$, one can write the energy density $\mathcal{E} = \langle \hat{H} \rangle_T / V$ as

$$\mathcal{E} = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{c\hbar k}{e^{\beta c\hbar k} - 1} = \frac{c\hbar}{\pi^2} \int_0^\infty dk \frac{k^3}{e^{\beta c\hbar k} - 1} \ , \tag{10}$$

where the factor 2 is due to the two possible polarizations ($s = 1, 2$). By using $\omega = ck$ instead of k as integration variable one gets

$$\mathcal{E} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} = \int_0^\infty d\omega \ \rho(\omega) \ , \tag{11}$$

where

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \tag{12}$$

is the energy density per frequency, i.e. the familiar formula of the black-body radiation, obtained for the first time in 1900 by Max Planck.

Thermal energy of photons (III)

The previous integral can be explicitly calculated and it gives

$$\mathcal{E} = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4 \,, \tag{13}$$

which is nothing but the Stefan-Boltzmann law. In an similar way one determines the average number density of photons:

$$n = \frac{\langle \hat{N} \rangle_T}{V} = \frac{1}{\pi^2 c^3} \int_0^\infty d\omega \, \frac{\omega^2}{e^{\beta\hbar\omega} - 1} = \frac{2\zeta(3) k_B^3}{\pi^2 c^3 \hbar^3} T^3 \,. \tag{14}$$

where $\zeta(3) \simeq 1.202$. Notice that both energy density $\mathcal{E}$ and number density n of photons go to zero as the temperature T goes to zero. We stress that these results are obtained at thermal equilibrium and under the condition of a vanishing chemical potential, meaning that the number of photons is not conserved when the temperature is varied.