Lesson 2 - Quantum Properties of Light
Unit 2.1 Fock states vs Coherent states

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Quantum electromagnetic field (I)

We have seen that the quantum Hamiltonian of the light can be then written as

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \left(\hat{N}_{\mathbf{k}s} + \frac{1}{2} \right) . \quad (1)$$

Here

$$\hat{N}_{\mathbf{k}s} = \hat{a}^{+}_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} \quad (2)$$

is the single-mode number operator, which counts the number of photons in the single-mode state $|\mathbf{k}s\rangle$, with $\mathbf{k}$ the wavevector and $s = 1, 2$ the polarization.

The quantum electric and magnetic fields can be obtained from the classical expressions. In this way we obtain

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \left[\hat{a}_{\mathbf{k}s}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - \hat{a}^{+}_{\mathbf{k}s}\, e^{-(i\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right] \varepsilon_{\mathbf{k}s} , \quad (3)$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k V}} \left[\hat{a}_{\mathbf{k}s}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - \hat{a}^{+}_{\mathbf{k}s}\, e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right] i\mathbf{k} \wedge \varepsilon_{\mathbf{k}s} \quad (4)$$

Quantum electromagnetic field (II)

Let us now consider for simplicity a linearly polarized monochromatic wave of the radiation field with wavevector **k** and polarization *s*.
One finds immediately that the quantum electric field can be then written in a simplified notation as

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, i \left[\hat{a}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^+\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \varepsilon \tag{5}$$

where $\omega = \omega_k = c|\mathbf{k}|$. Notice that, to simplify the notation, we have removed the subscripts in the annihilation and creation operators $\hat{a}$ and $\hat{a}^+$.
If there are exactly n photons in this polarized monochromatic wave the Fock state of the system is given by

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^+\right)^n |0\rangle \ . \tag{6}$$

Quantum electromagnetic field (III)

It is then straightforward to show that

$$\langle n|\hat{\mathbf{E}}(\mathbf{r},t)|n\rangle = \mathbf{0}, \tag{7}$$

for all values of the photon number n, no matter how large. This result holds for all modes, which means then that the expectation value of the electric field in any many-photon Fock state is zero.
On the other hand, the expectation value of $\hat{E}(\mathbf{r},t)^2$ is given by

$$\langle n|\hat{E}(\mathbf{r},t)^2|n\rangle = \frac{\hbar\omega}{\varepsilon_0 V}\left(n+\frac{1}{2}\right) . \tag{8}$$

Obviously a similar reasoning applies for the magnetic field (4).

Coherent states (I)

The strange result of Eq. (7) is due to the fact that the expectation value is performed with the Fock state $|n\rangle$, which means that the number of photons is fixed because

$$\hat{N}|n\rangle = n|n\rangle \; . \tag{9}$$

Nevertheless, usually the number of photons in the radiation field is not fixed, in other words the system is not in a pure Fock state. For example, the radiation field of a well-stabilized laser device operating in a single mode is described by a coherent state $|\alpha\rangle$, such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \; , \tag{10}$$

with

$$\langle\alpha|\alpha\rangle = 1 \; . \tag{11}$$

Coherent states (II)

The coherent state $|\alpha\rangle$, introduced in 1963 by Roy Glauber, is thus the eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue $\alpha = |\alpha| e^{i\theta}$. $|\alpha\rangle$ does not have a fixed number of photons, i.e. it is not an eigenstate of the number operator $\hat{N}$, and it is not difficult to show that $|\alpha\rangle$ can be expanded in terms of number (Fock) states $|n\rangle$ as follows

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ . \tag{12}$$

From Eq. (10) one immediately finds

$$\bar{N} = \langle\alpha|\hat{N}|\alpha\rangle = |\alpha|^2 \ , \tag{13}$$

and it is natural to set

$$\alpha = \sqrt{\bar{N}}\, e^{i\theta} \ , \tag{14}$$

where $\bar{N}$ is the average number of photons in the coherent state, while θ is the phase of the coherent state.

Coherent states (III)

We observe that the coherent state $|\alpha\rangle$ is such that

$$\langle\alpha|\hat{N}^2|\alpha\rangle = |\alpha|^2 + |\alpha|^4 = \bar{N} + \bar{N}^2 \tag{15}$$

and consequently

$$\langle\alpha|\hat{N}^2|\alpha\rangle - \langle\alpha|\hat{N}|\alpha\rangle^2 = \bar{N} \ , \tag{16}$$

while

$$\langle n|\hat{N}^2|n\rangle = n^2 \tag{17}$$

and consequently

$$\langle n|\hat{N}^2|n\rangle - \langle n|\hat{N}|n\rangle^2 = 0 \ . \tag{18}$$

Coherent states (IV)

It is now easy to prove that the expectation value of the electric field $\hat{\mathbf{E}}(\mathbf{r}, t)$ of the linearly polarized monochromatic wave, Eq. (5), in the coherent state $|\alpha\rangle$ reads

$$\langle\alpha|\hat{\mathbf{E}}(\mathbf{r}, t)|\alpha\rangle = -\sqrt{\frac{2\bar{N}\hbar\omega}{\varepsilon_0 V}} \sin(\mathbf{k}\cdot\mathbf{r} - \omega t + \theta)\,\boldsymbol{\varepsilon} \,, \tag{19}$$

while the expectation value of $\hat{E}(\mathbf{r}, t)^2$ is given by

$$\langle\alpha|\hat{E}(\mathbf{r}, t)^2|\alpha\rangle = \frac{2\bar{N}\hbar\omega}{\varepsilon_0 V} \sin^2(\mathbf{k}\cdot\mathbf{r} - \omega t + \theta) + \frac{\hbar\omega}{2\varepsilon_0 V} \,. \tag{20}$$

These results suggest that the coherent state is indeed a useful tool to investigate the correspondence between quantum field theory and classical field theory.