# Lesson 1 - Second Quantization of Light
# Unit 1.2 Quantization of the electromagnetic field

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# Classical energy and Fourier expansion (I)

A familiar result of electromagnetism is that the classical energy of the electromagnetic field in vacuum is given by

$$H = \int d^3\mathbf{r} \, \left( \frac{\varepsilon_0}{2} \mathbf{E}(\mathbf{r},t)^2 + \frac{1}{2\mu_0} \mathbf{B}(\mathbf{r},t)^2 \right) \, . \tag{1}$$

Remember that in the Coulomb gauge we have

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \, , \tag{2}$$

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \, , \tag{3}$$

and consequently

$$H = \int d^3\mathbf{r} \left( \frac{\varepsilon_0}{2} \left( \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \right)^2 + \frac{1}{2\mu_0} \left( \boldsymbol{\nabla} \wedge \mathbf{A}(\mathbf{r},t) \right)^2 \right) \tag{4}$$

# Classical energy and Fourier expansion (II)

We now expand the vector potential $\mathbf{A}(\mathbf{r}, t)$ as a Fourier series of monochromatic plane waves:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{s} \left[ A_{\mathbf{k}s}(t) \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} + A^*_{\mathbf{k}s}(t) \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} \right] \boldsymbol{\varepsilon}_{\mathbf{k}s} \ , \tag{5}$$

where

$$A_{\mathbf{k}s}(t) = A_{\mathbf{k}s}(0) \ e^{-i\omega_k t} \tag{6}$$

and

$$A^*_{\mathbf{k}s}(t) = A^*_{\mathbf{k}s}(0) \ e^{i\omega_k t} \tag{7}$$

are the dimensional complex conjugate coefficients of the expansion, the complex plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$ normalized in a volume $V$ are the basis functions of the expansion, and $\boldsymbol{\varepsilon}_{\mathbf{k}1}$ and $\boldsymbol{\varepsilon}_{\mathbf{k}2}$ are two mutually orthogonal real unit vectors of polarization which are also orthogonal to $\mathbf{k}$.
Inserting this Fourier expansion into the total energy $H$ of the electromagnetic field we get

$$H = \sum_{\mathbf{k}} \sum_{s} \epsilon_0 \omega_k^2 \ (A^*_{\mathbf{k}s} A_{\mathbf{k}s} + A_{\mathbf{k}s} A^*_{\mathbf{k}s}) \ . \tag{8}$$

# Classical energy and Fourier expansion (III)

It is now convenient to introduce dimensionless complex coefficients $a_{\mathbf{k}s}(t)$ and $a^*_{\mathbf{k}s}(t)$ related to the dimensional complex coefficients $A_{\mathbf{k}s}(t)$ and $A^*_{\mathbf{k}s}(t)$ by

$$A_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k}}\, a_{\mathbf{k}s}(t) \ . \tag{9}$$

$$A^*_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k}}\, a^*_{\mathbf{k}s}(t) \ . \tag{10}$$

In this way the energy $H$ reads

$$H = \sum_{\mathbf{k}} \sum_{s} \frac{\hbar\omega_k}{2} \left( a^*_{\mathbf{k}s} a_{\mathbf{k}s} + a_{\mathbf{k}s} a^*_{\mathbf{k}s} \right) \ . \tag{11}$$

Instead of using the complex amplitudes $a^*_{\mathbf{k}s}(t)$ and $a_{\mathbf{k}s}(t)$ one can introduce the real variables

$$q_{\mathbf{k}s}(t) = \sqrt{\frac{2\hbar}{\omega_k}}\,\frac{1}{2} \left( a_{\mathbf{k}s}(t) + a^*_{\mathbf{k}s}(t) \right) \tag{12}$$

$$p_{\mathbf{k}s}(t) = \sqrt{2\hbar\omega_k}\,\frac{1}{2i} \left( a_{\mathbf{k}s}(t) - a^*_{\mathbf{k}s}(t) \right) \ . \tag{13}$$

# Quantization of the single modes (I)

In this way the energy of the radiation field reads

$$H = \sum_{\mathbf{k}} \sum_{s} \left( \frac{p_{\mathbf{k},s}^2}{2} + \frac{1}{2}\omega_k^2 \, q_{\mathbf{k}s}^2 \right) \ . \tag{14}$$

This energy resembles that of infinitely many harmonic oscillators with unitary mass and frequency $\omega_k$.
In 1927 Paul Dirac performed the quantization of the classical Hamiltonian (14) by promoting the real coordinates $q_{\mathbf{k}s}$ and the real momenta $p_{\mathbf{k}s}$ to operators:

$$q_{\mathbf{k}s} \to \hat{q}_{\mathbf{k}s} \ , \tag{15}$$

$$p_{\mathbf{k}s} \to \hat{p}_{\mathbf{k}s} \ , \tag{16}$$

satisfying the commutation relations

$$[\hat{q}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s'}] = i\hbar \, \delta_{\mathbf{k},\mathbf{k}'} \, \delta_{s,s'} \ , \tag{17}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

# Quantization of the single modes (II)

The quantum Hamiltonian is thus given by

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \left( \frac{\hat{p}^2_{\mathbf{k},s}}{2} + \frac{1}{2}\omega_k^2 \, \hat{q}^2_{\mathbf{k}s} \right) \; . \tag{18}$$

Following a standard approach for the canonical quantization of the Harmonic oscillator, we introduce annihilation and creation operators

$$\hat{a}_{\mathbf{k}s} = \sqrt{\frac{\omega_k}{2\hbar}} \left( \hat{q}_{\mathbf{k}s} + \frac{i}{\omega_k} \, \hat{p}_{\mathbf{k}s} \right) \; , \tag{19}$$

$$\hat{a}^{+}_{\mathbf{k}s} = \sqrt{\frac{\omega_k}{2\hbar}} \left( \hat{q}_{\mathbf{k}s} - \frac{i}{\omega_k} \, \hat{p}_{\mathbf{k}s} \right) \; , \tag{20}$$

which satisfy the commutation relations

$$[\hat{a}_{\mathbf{k}s}, \hat{a}^{+}_{\mathbf{k}'s'}] = \delta_{\mathbf{k},\mathbf{k}'} \, \delta_{s,s'} \; , \tag{21}$$

and the quantum Hamiltonian (18) becomes

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \left( \hat{a}^{+}_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} + \frac{1}{2} \right) \; . \tag{22}$$

# Ladder operators and their properties (I)

The operators $\hat{a}_{\mathbf{k}s}$ and $\hat{a}^+_{\mathbf{k}s}$ act in the Fock space $\mathcal{F}$, i.e. the infinite dimensional Hilbert space of "number representation" introduced in 1932 by Vladimir Fock. A generic state of this Fock space $\mathcal{F}$ is given by

$$|\dots n_{\mathbf{k}s} \dots n_{\mathbf{k}'s'} \dots n_{\mathbf{k}''s''} \dots \rangle \ , \tag{23}$$

meaning that there are $n_{\mathbf{k}s}$ photons with wavevector $\mathbf{k}$ and polarization $s$, $n_{\mathbf{k}'s'}$ photons with wavevector $\mathbf{k}'$ and polarization $s'$, $n_{\mathbf{k}''s''}$ photons with wavevector $\mathbf{k}''$ and polarization $s''$, et cetera.
Notice that in the definition of the Fock space $\mathcal{F}$ one must include the space of 0 photons, containing only the vacuum state

$$|0\rangle = |\dots 0 \dots 0 \dots 0 \dots \rangle \ . \tag{24}$$

The operators $\hat{a}_{\mathbf{k}s}$ and $\hat{a}^+_{\mathbf{k}s}$ are called ladder operators, or annihilation and creation operators, because they respectively destroy and create one photon with wavevector $\mathbf{k}$ and polarization $s$, namely

$$\hat{a}_{\mathbf{k}s}|\dots n_{\mathbf{k}s} \dots \rangle = \sqrt{n_{\mathbf{k}s}}\ |\dots n_{\mathbf{k}s} - 1 \dots \rangle \ , \tag{25}$$

$$\hat{a}^+_{\mathbf{k}s}|\dots n_{\mathbf{k}s} \dots \rangle = \sqrt{n_{\mathbf{k}s} + 1}\ |\dots n_{\mathbf{k}s} + 1 \dots \rangle \ . \tag{26}$$

# Ladder operators and their properties (II)

From Eqs. (25) and (26) it follows immediately that

$$\hat{N}_{\mathbf{k}s} = \hat{a}^{+}_{\mathbf{k}s}\hat{a}_{\mathbf{k}s} \tag{27}$$

is the number operator which counts the number of photons in the single-particle state $|\mathbf{k}s\rangle$, i.e.

$$\hat{N}_{\mathbf{k}s}|\ldots n_{\mathbf{k}s}\ldots\rangle = n_{\mathbf{k}s}\ |\ldots n_{\mathbf{k}s}\ldots\rangle\ . \tag{28}$$

The quantum Hamiltonian of the light can be then written as

$$\hat{H} = \sum_{\mathbf{k}}\sum_{s}\hbar\omega_k\left(\hat{N}_{\mathbf{k}s} + \frac{1}{2}\right)\ . \tag{29}$$

In the case of the quantum electromagnetic field there is an infinite number of quantum harmonic oscillators and the total zero-point energy is given by

$$E_{vac} = \sum_{\mathbf{k}}\sum_{s}\frac{1}{2}\hbar\omega_k\ . \tag{30}$$

This infinite constant $E_{vac}$ is usually eliminated by simply shifting to zero the energy associated to the vacuum state $|0\rangle$.