Lesson 1 - Second Quantization of Light
Unit 1.1 Electromagnetic waves

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Maxwell equations (I)

The light is an electromagnetic field characterized by the coexisting presence of an electric field $\mathbf{E}(\mathbf{r}, t)$ and a magnetic field $\mathbf{B}(\mathbf{r}, t)$. From the equations of James Clerk Maxwell in vacuum and in the absence of sources, given by

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 , \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 , \tag{2}$$

$$\boldsymbol{\nabla} \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} , \tag{3}$$

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} , \tag{4}$$

one finds that the coupled electric and magnetic fields satisfy the d'Alambert wave equations

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{E}(\mathbf{r}, t) = \mathbf{0} , \tag{5}$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{B}(\mathbf{r}, t) = \mathbf{0} , \tag{6}$$

Maxwell equations (II)

where

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \cdot 10^8 \text{ m/s} \tag{7}$$

is the speed of light in the vacuum.
Equations (5) and (6), which are fully confirmed by experiments, admit monochromatic complex plane wave solutions

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \ , \tag{8}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \ , \tag{9}$$

where $\mathbf{k}$ is the wavevector and ω the angular frequency, such that

$$\omega = c\, k \ , \tag{10}$$

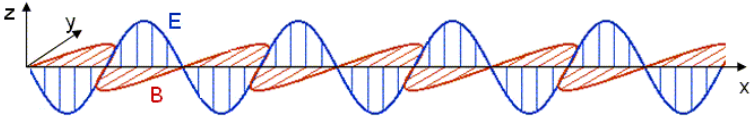
is the dispersion relation, with $k = |\mathbf{k}|$ is the wavenumber.

Maxwell equations (III)

From Maxwell's equations one finds that the vectors **E** and **B** are mutually orthogonal and such that

$$E = c\,B\ , \tag{11}$$

where $E = |\mathbf{E}|$ and $B = |\mathbf{B}|$. In addition they are transverse fields, i.e. orthogonal to the wavevector **k**, which gives the direction of propagation of the wave.



For completeness, let us remind that the wavelength λ is given by

$$\lambda = \frac{2\pi}{k}\ , \tag{12}$$

and that the linear frequency ν and the angular frequency $\omega = 2\pi\nu$ are related to the wavelength λ and to the wavenumber k by the formulas

$$\lambda\,\nu = \frac{\omega}{k} = c\ . \tag{13}$$

Electromagnetic potentials and Coulomb gauge (I)

In full generality the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ can be expressed in terms of a scalar potential $\phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$ as follows

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} , \quad (14)$$

$$\mathbf{B} = \nabla \wedge \mathbf{A} . \quad (15)$$

Actually these equations do not determine the electromagnetic potentials uniquely, since for an arbitrary scalar function $\Lambda(\mathbf{r}, t)$ the so-called "gauge transformation"

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t} , \quad (16)$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\Lambda , \quad (17)$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unaltered.

Electromagnetic potentials and Coulomb gauge (II)

We use this remarkable property to choose a gauge transformation such that

$$\nabla \cdot \mathbf{A} = 0 \ . \tag{18}$$

This condition defines the Coulomb (or radiation) gauge, and the vector field $\mathbf{A}$ is called transverse field.

For a complex monochromatic plane wave

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{19}$$

the Coulomb gauge (18) gives

$$\mathbf{k} \cdot \mathbf{A} = 0 \ , \tag{20}$$

i.e. $\mathbf{A}$ is perpendicular (transverse) to the wavevector $\mathbf{k}$. In the vacuum and without sources, from the first Maxwell equation (1) and Eq. (14) one immediately finds

$$\nabla^2 \phi + \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \mathbf{A}) = 0 \ , \tag{21}$$

and under the Coulomb gauge (18) one gets

$$\nabla^2 \phi = 0 \ . \tag{22}$$

Electromagnetic potentials and Coulomb gauge (III)

Imposing that the scalar potential is zero at infinity, this Laplace's equation has the unique solution

$$\phi(\mathbf{r}, t) = 0 \,, \tag{23}$$

and consequently

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \,, \tag{24}$$

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \,. \tag{25}$$

Thus, in the Coulomb gauge one needs only the electromagnetic vector potential $\mathbf{A}(\mathbf{r}, t)$ to obtain the electromagnetic field if there are no charges and no currents.
The electromagnetic field described by these equations is often called the radiation field, and also the vector potential satisfies the wave equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A}(\mathbf{r}, t) = \mathbf{0} \,. \tag{26}$$