ANALISI MATEMATICA 1
Information Engineering

23.01.2023

TEMA 1

Exercise 1 (score 9) Consider the function

$$f(x) = \sqrt{x^2 + x} - x$$

(a) Find the maximal domain of f;

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : x^2 + x \geq 0\} = (-\infty, -1] \cup [0, +\infty).$$

(b) compute the limits at significative points and asymptotes

f is continuous at every $x \in \mathrm{Dom}(f)$, so that

$$\lim_{x\to -1^-} f(x) = f(-1) = 1, \qquad \lim_{x\to 0^+} f(x) = f(0) = 0;$$

moreover

$$\lim_{x\to -\infty} f(x) = +\infty$$

and

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} f(x) \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim_{x\to +\infty} \frac{x}{\sqrt{x^2+x}+x} = \frac{1}{2},$$

in particular, $y = \frac{1}{2}$ is a horizontal asymptote for $x \to +\infty$. Let us compute the asymptote for $x \to -\infty$:

$$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} -\sqrt{1 + \frac{1}{x}} - 1 = -2$$

and

$$\lim_{x\to -\infty} f(x) + 2x = \lim_{x\to -\infty} \left(\sqrt{x^2+x} + x\right) \frac{\sqrt{x^2+x} - x}{\sqrt{x^2+x} - x} = \lim_{x\to -\infty} \frac{x}{\sqrt{x^2+x} - x} = -\frac{1}{2}.$$

Hence $y = -2x - \frac{1}{2}$ is asymptote for $x \to -\infty$.

(c) Study the differentiability of f, compute the derivative and possible limits of the derivative; discuss the monotonicity of f, determine if f is upper [lower] bounded, and in the positive case find the supremum [infimum], and relative and absolute minima and maxima ;

f is differentiable in $(-\infty, -1) \cup (0, +\infty)$ because it is a composition and sum of of differentiable functions ; in particular, $x^2 + x > 0$ in $(-\infty, -1) \cup (0, +\infty)$ and $\sqrt{y}$ is differentiable for $y > 0$. On those points the derivative is

$$f'(x) = \frac{2x+1}{2\sqrt{x^2+x}} - 1.$$

As for the limits, one has

$$, \quad \lim_{x\to -1^-} f'(x) = -\infty, \quad \lim_{x\to 0^+} f'(x) = +\infty,$$

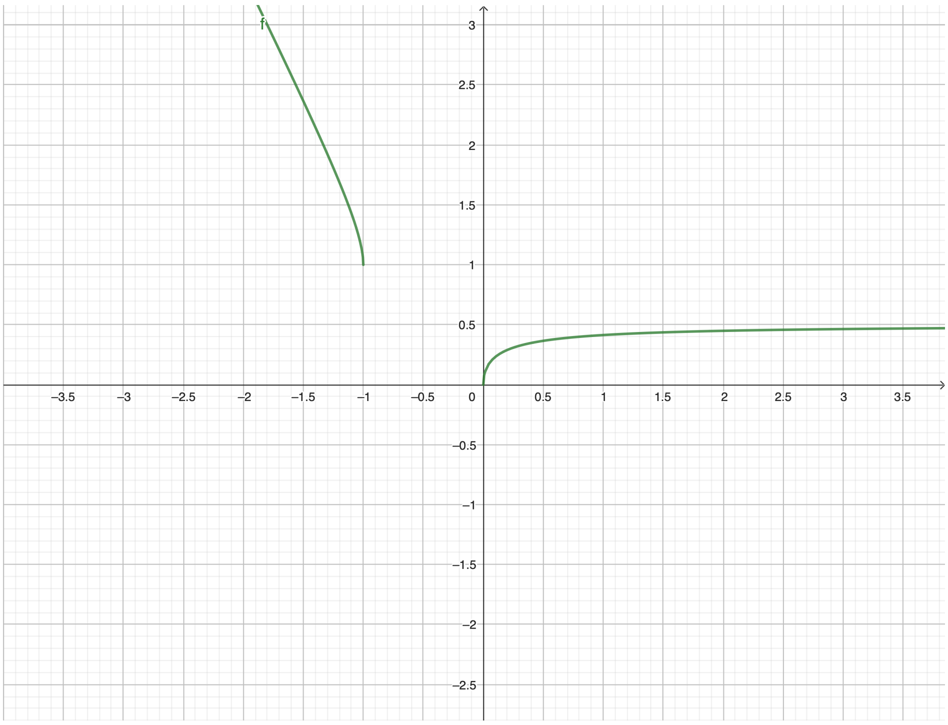

Figure 1: Graph di f

In particular, f is not differentiable at $x = -1$ and $x = 0$. For every $x \in (-\infty, -1) \cup (0, +\infty)$ we have

$$f'(x) = 0 \Leftrightarrow 2x + 1 = 2\sqrt{x^2 + x}$$

and there are no solutions to this equation because $(2x+1)^2 = 4x^2 + 4x + 1 \neq 4x^2 + 4x = (2\sqrt{x^2 + x})^2$. Moreover $f'(x) < 0$ for every $x \in (-\infty, -1)$ and $f'(x) > 0$ for every $x \in (0, +\infty)$, hence f is decreasing in $(-\infty, -1]$ and increasing in $[0, +\infty)$; moreover -1 and 0 are relative minimum points, 0 is an absolute minimum point and $f(0) = 0$ is the infimum (actually minimum) di f.There are no relative maximum points and the supremum is $+\infty$.

(d) plot a qualitative graph of f.

See picture 1.

Exercise 2 (score 7) Consider the complex polynomial equation:

$$z^3 + \alpha z^2 + iz = -\alpha i \qquad (\alpha \in \mathbb{R})$$

(a) Determine the value of the parameter α such that this equation has $z_0 := 4$ as a solution;

By imposing that the equation holds true for $z = 4$ we get $64 + 16\alpha + 4i = -\alpha i$ hence $\alpha = -\frac{64+4i}{16+i} = -4\frac{16+i}{16+i} = -4$.

(b) If α is as in pont a), find the remaining solutions of the equation.

We have $z^3 - 4z^2 + iz - 4i = (z-4)(z^2 + i)$ hence the remaining solutions are the square roots of di $-i$: for compute them, observe that $|-i| = 1$ and $\mathrm{Arg}(-i) = -\frac{\pi}{2}$. Hence z_1 and z_2 have modulus 1 and argument $-\frac{\pi}{4}$ and $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ respectively, that is

$$z_1 = e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$
$$z_2 = e^{i\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

Another path to get the same result would have consisted in first observing that $z^3 + \alpha z^2 + iz + \alpha i = (z^2+i)(z+\alpha)$ and then concluding that the solutions are $-\alpha, z_1, z_2$, where the roots z_1 and z_2 are computed as in point (b).

Exercise 3 (score 8) (a) Compute the limit

$$\lim_{x\to 0^+} [1 - \arcsin x]^{\frac{1}{x}}.$$

Since the exponential function is continuous, we have

$$\lim_{x\to 0^+} [1 - \arcsin(x)]^{\frac{1}{x}} = e^{\lim_{x\to 0^+} \frac{\log(1-\arcsin(x))}{x}} = \frac{1}{e},$$

because

$$\lim_{x\to 0^+} \frac{\log(1-\arcsin(x))}{x} = \lim_{x\to 0^+} \frac{\log(1-(x+o(x))}{x} = \lim_{x\to 0^+} \frac{-(x+o(x)) + o\big(-x-o(x)\big)}{x} =$$

$$\lim_{x\to 0^+} \frac{-x+o\big(x\big)}{x} = -1$$

(b) Study the character of the seris

$$\sum_{n=1}^{+\infty} \left[1 - \arcsin\left(\frac{1}{n}\right)\right]^{n^2}.$$

We begin by observing that the terms of this series are positive for every $n \geq 1$. Therefore we can apply the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left[1 - \arcsin\left(\frac{1}{n}\right)\right]^{n^2}} = \lim_{n\to\infty} \left[1 - \arcsin\left(\frac{1}{n}\right)\right]^{n} = \frac{1}{e}$$

because, by the change of variable $x = \frac{1}{n}$ and by point (a), we get

$$\lim_{n\to\infty} \left[1 - \arcsin\left(\frac{1}{n}\right)\right]^{n} = \lim_{x\to 0^+} [1 - \arcsin(x)]^{\frac{1}{x}} = \frac{1}{e} < 1,$$

so the series is convergent.

Exercise 4 (score 8) Consider the family of functions

$$f_\alpha(x) = (x-2)\arctan(x^\alpha) \qquad \alpha \in \mathbb{R}$$

(a) Compute

$$\int f_1(x)\,dx$$

$$\int f_1(x)\,dx = \int \big[(x-2)\arctan x\big]\,dx = \frac{(x-2)^2}{2}\arctan x - \int \left[\frac{(x-2)^2}{2(1+x^2)}\right] dx + c \qquad c \in \mathbb{R}$$

Now

$$\frac{(x-2)^2}{2(1+x^2)} = \frac{x^2+1}{2(x^2+1)} + \frac{3}{2(x^2+1)} - \frac{4x}{2(x^2+1)}$$

so that

$$\int f_1(x)\,dx = \frac{(x-2)^2-3}{2}\arctan x - \frac{1}{2}x + \log(x^2+1) + c \qquad c \in \mathbb{R}$$

(b)Determine for which values of the parameter $\alpha \in \mathbb{R}$, the integral

$$\int_1^{+\infty} f_\alpha(x)\,dx$$

is convergent.

There is no problem at the extreme 1, for f_α is continuous at every $x \geq 1$. Hence we have to compute

$$\lim_{k\to+\infty}\int_1^k f_\alpha(x)\,dx = \lim_{k\to+\infty}\int_1^k \big((x-2)\arctan(x^\alpha)\big)\,dx$$

If $\alpha \geq 0$, one has $f_\alpha \sim (x-2)$ for $x \to +\infty$, so, by the asymptotic comparison test, the integral is not convergent.

If instead $\alpha < 0$, by $\arctan y = y + o(y)$ one has $f_\alpha \sim \frac{1}{x^{-1-\alpha}}$, so by the asymptotic comparison test, the integral is convergent if and only if $-1-\alpha > 1$, i.and., if and only if $\alpha < -2$

Some Taylor expansions:

$$\begin{aligned}
\arcsin(x) &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + o(x^6),\\
\arctan(x) &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \qquad \forall n \geq 0
\end{aligned}$$

CALCULUS 1

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23.01.2023

TEMA 2

Exercise 1 (score 9) Consider the function

$$f(x) = x - \sqrt{x^2 - x}$$

(a) find the domain of f;

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : x^2 - x \geq 0\} = (-\infty, 0] \cup [1, +\infty).$$

(b) compute the limits at significative points and asymptotes

f is continuous for every $x \in \mathrm{Dom}(f)$ hence

$$\lim_{x\to 0^-} f(x) = f(0) = 0, \qquad \lim_{x\to 1^+} f(x) = f(1) = 1;$$

moreover

$$\lim_{x\to -\infty} f(x) = -\infty$$

and

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} f(x)\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} = \lim_{x\to +\infty} \frac{x}{x + \sqrt{x^2 - x}} = \frac{1}{2},$$

in particular, $y = \frac{1}{2}$ is horizontal asymptote for $x \to +\infty$. Let us compute the asymptote for $x \to -\infty$:

$$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} 1 + \sqrt{1 - \frac{1}{x}} = 2$$

and

$$\lim_{x\to -\infty} f(x) - 2x = \lim_{x\to -\infty} -\left(x + \sqrt{x^2 - x}\right)\frac{x - \sqrt{x^2 - x}}{x - \sqrt{x^2 - x}} = \lim_{x\to -\infty} -\frac{x}{x - \sqrt{x^2 - x}} = -\frac{1}{2}.$$

Hence $y = 2x - \frac{1}{2}$ is asymptote for $x \to -\infty$.

(c) study the differentiability of f; compute the derivative and its limits (at non-interior points); discuss the monotonicity of f and determine the infimum and the supremum di f ed eventuali points of minimo and maximum relativo ed assoluto;

f is differentiable in $(-\infty, 0) \cup (1, +\infty)$ because it is a composition and sum of differentiable functions ; in particular, $x^2 - x > 0$ in $(-\infty, 0) \cup (1, +\infty)$ and $\sqrt{y}$ is differentiable for $y > 0$. At these points the derivative is

$$f'(x) = 1 - \frac{2x - 1}{2\sqrt{x^2 - x}}.$$

As for the limits of the derivative, one has

$$\lim_{x\to 0^-} f'(x) = +\infty, \quad \lim_{x\to 1^+} f'(x) = -\infty,$$

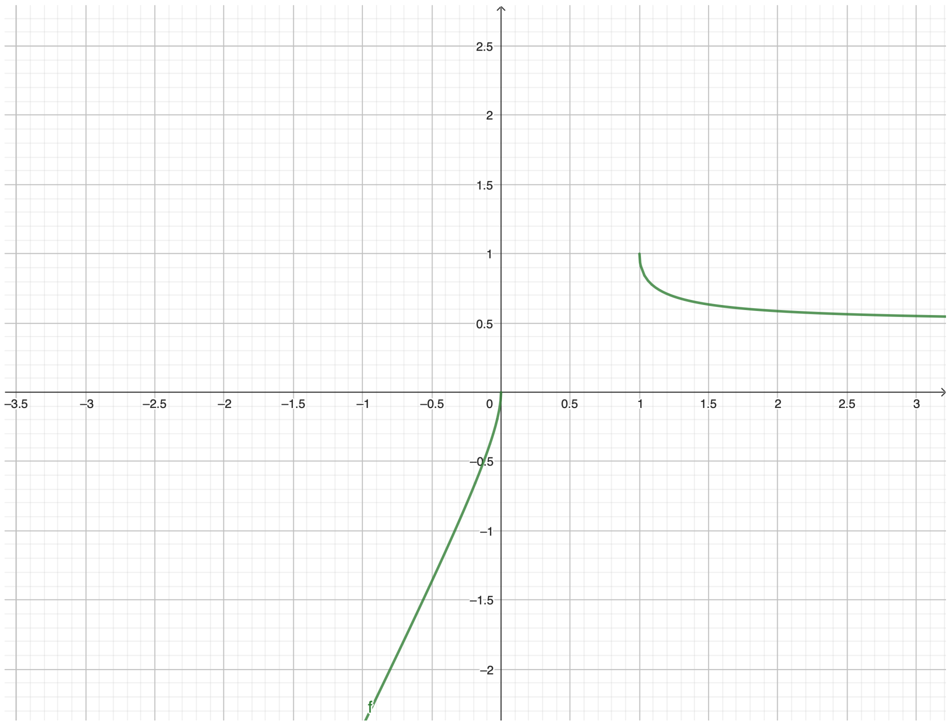

Figure 2: Graph di f

In particular, f is not differentiable nei points $x = 0$ and $x = 1$. For every $x \in (-\infty, 0) \cup (1, +\infty)$ we have

$$f'(x) = 0 \Leftrightarrow 2x - 1 = 2\sqrt{x^2 - x}$$

and there are no solutions to this equation because$(2x-1)^2 = 4x^2 - 4x + 1 \neq 4x^2 - 4x = (2\sqrt{x^2 - x})^2$. Moreover $f'(x) > 0$ for every $x \in (-\infty, 0)$ and $f'(x) < 0$ for every $x \in (1, +\infty)$, hence f is increasing in $(-\infty, 0]$ and decreasing in $[1, +\infty)$; moreover 0 and 1 are relative maximum points, 1 is an absolute maximum point, and $f(1) = 1$ is the supremum (maximum) di f.The function is not lower bounded.

(d) plot a qualitative graph of f.

See picture 2.

Exercise 2 (score 7) Consider the complex polynomial equation:

$$z^3 + \alpha z^2 + iz = -\alpha i \qquad (\alpha \in \mathbb{R})$$

(a) Determine the value of the parameter α such that this equation has $z_0 := -4$ as a solution;

By imposing that the equation holds true for $z = -4$ we get $-64 + 16\alpha - 4i = -\alpha i$ hence $\alpha = \frac{64+4i}{16+i} = 4\frac{16+i}{16+i} = 4$.

(b) Find the remaining roots

We have $z^3 + 4z^2 + iz + 4i = (z+4)(z^2 + i)$ hence the remaining solutions are the square roots of $-i$: since $|-i| = 1$ and $\text{Arg}(-i) = -\frac{\pi}{2}$ the roots z_1 and z_2 have modulus 1 and argument $-\frac{\pi}{4}$ and $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ respectively, that is

$$z_1 = e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$
$$z_2 = e^{i\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

As an alternative, we might have observed that $z^3 + \alpha z^2 + iz + \alpha i = (z^2 + i)(z + \alpha)$, so the solutions are $-\alpha, z_1, z_2$: the roots z_1 and z_2 are as in (b) and for point (a) to be verified it is necessary and suffic $-\alpha = -4$. If α is as in pont a), find the remaining solutions of the equation.

Exercise 3 (score 8) (a) Compute the limit

$$\lim_{x\to 0^+} [1 - \sinh(x)]^{\frac{1}{x}}.$$

Since the exponential function is continuous, we have

$$\lim_{x\to 0^+} [1 - \sinh(x)]^{\frac{1}{x}} = e^{\lim_{x\to 0^+} \frac{\log(1-\sinh(x))}{x}} = \frac{1}{e},$$

because

$$\lim_{x\to 0^+} \frac{\log(1 - \sinh(x))}{x} = \lim_{x\to 0^+} \frac{\log(1 - (x + o(x))}{x} = \lim_{x\to 0^+} \frac{-(x + o(x)) + o\big(-x - o(x)\big)}{x} =$$

$$\lim_{x\to 0^+} \frac{-x + o(x)}{x} = -1$$

(b) Study the character of the seris

$$\sum_{n=1}^{+\infty} \left[1 - \sinh\left(\frac{1}{n}\right)\right]^{n^2}.$$

We begin by observing that the terms of this series are positive for every $n \geq 2$. Therefore we can apply the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left[1 - \sinh\left(\frac{1}{n}\right)\right]^{n^2}} = \lim_{n\to\infty} \left[1 - \sinh\left(\frac{1}{n}\right)\right]^{n} = \frac{1}{e}$$

because, by the change of variable $x = \frac{1}{n}$ and by point (a), we get

$$\lim_{n\to\infty} \left[1 - \sinh\left(\frac{1}{n}\right)\right]^{n} = \lim_{x\to 0^+} [1 - \sinh(x)]^{\frac{1}{x}} = \frac{1}{e} < 1,$$

so the series is convergent.

Exercise 4 (score 8) Consider the family of functions

$$f_\alpha(x) = (x + 1)\arctan(x^\alpha)$$

(a) Compute

$$\int f_1(x)\, dx$$

$$\int f_1(x)\, dx = \int \big[(x + 1)\arctan x\big]\, dx = \frac{(x + 1)^2}{2} \arctan x - \int \left[\frac{(x + 1)^2}{2(1 + x^2)}\right] dx + c \qquad c \in \mathbb{R}$$

Now

$$\frac{(x + 1)^2}{2(1 + x^2)} = \frac{x^2 + 1}{2(x^2 + 1)} + \frac{2x}{2(x^2 + 1)}$$

so that

$$\int f_1(x)\,dx = \frac{(x+1)^2}{2}\arctan x - \frac{1}{2}x - \frac{1}{2}\log((x^2+1)) + c \qquad c \in \mathbb{R}$$

(b)Determine for which values of the parameter $\alpha \in \mathbb{R}$, the integral

$$\int_1^{+\infty} f_\alpha(x)\,dx$$

is convergent.

There is no problem at the extreme 1, for f_α is continuous at every $x \geq 1$. Hence we have to compute

$$\lim_{k\to+\infty}\int_1^k f_\alpha(x)\,dx = \lim_{k\to+\infty}\int_1^k \big((x+1)\arctan(x^\alpha)\big)\,dx$$

If $\alpha \geq 0$, one has $f_\alpha \sim (x+1)$ for $x \to +\infty$, so, by the asymptotic comparison test, the integral is not convergent.

If instead $\alpha < 0$, by $\arctan y = y + o(y)$ one has $f_\alpha \sim \frac{1}{x^{-1-\alpha}}$, so by the asymptotic comparison test, the integral is convergent if and only if $-1-\alpha > 1$, i.and., if and only if $\alpha < -2$

Tempo: due ore and mezza (comprensive of domande of teoria). Viene corretto solo ciò che ànd scritto sul foglio intestato. È vietato tenere libri, appunti, telefoni and calcolatrici di qualsiasi tipo.

Some Taylor expansions:

$$\sinh(x) = x + \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}),$$
$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \qquad \forall n \geq 0$$

CALCULUS 1
Information Engineering

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TEMA 3

Exercise 1 (score 9) Consider the function

$$f(x) = \sqrt{x^2 + 2x} - x$$

(a) determinare il dominio di f ed eventuali simmetrie (is not richiesto lo studio del segno);

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : x^2 + 2x \geq 0\} = (-\infty, -2] \cup [0, +\infty).$$

(b) compute the limits at significative points and asymptotes

f is continuous for every $x \in \mathrm{Dom}(f)$ hence

$$\lim_{x\to -2^-} f(x) = f(-2) = 2, \qquad \lim_{x\to 0^+} f(x) = f(0) = 0;$$

moreover

$$\lim_{x\to -\infty} f(x) = +\infty$$

and

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} f(x) \frac{\sqrt{x^2 + 2x} + x}{\sqrt{x^2 + 2x} + x} = \lim_{x\to +\infty} \frac{2x}{\sqrt{x^2 + 2x} + x} = 1,$$

in particular, $y = 1$ is horizontal asymptote for $x \to +\infty$. Let us compute the asymptote for $x \to -\infty$:

$$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} -\sqrt{1 + \frac{2}{x}} - 1 = -2$$

and

$$\lim_{x\to -\infty} f(x) + 2x = \lim_{x\to -\infty} \left(\sqrt{x^2 + 2x} + x\right) \frac{\sqrt{x^2 + 2x} - x}{\sqrt{x^2 + 2x} - x} = \lim_{x\to -\infty} \frac{2x}{\sqrt{x^2 + x} - x} = -1.$$

Hence $y = -2x - 1$ is asymptote for $x \to -\infty$.

(c) studiare la derivabilità di f nel suo dominio, calcolare la derivata prima ed eventuali limiti della derivata, ove necessario; discuss the monotonicity of f and determine the infimum and the supremum di f ed relative or absolute minimum and maximum points;

f is differentiable on $(-\infty, -2) \cup (0, +\infty)$ because it is a composition and sum of differentiable functions ; in particular, $x^2 + 2x > 0$ in $(-\infty, -2) \cup (0, +\infty)$ and $\sqrt{y}$ is differentiable for $y > 0$. The derivative is

$$f'(x) = \frac{x + 1}{\sqrt{x^2 + 2x}} - 1.$$

As for the derivative limits, one has:

$$\lim_{x\to -2^-} f'(x) = -\infty, \quad \lim_{x\to 0^+} f'(x) = +\infty,$$

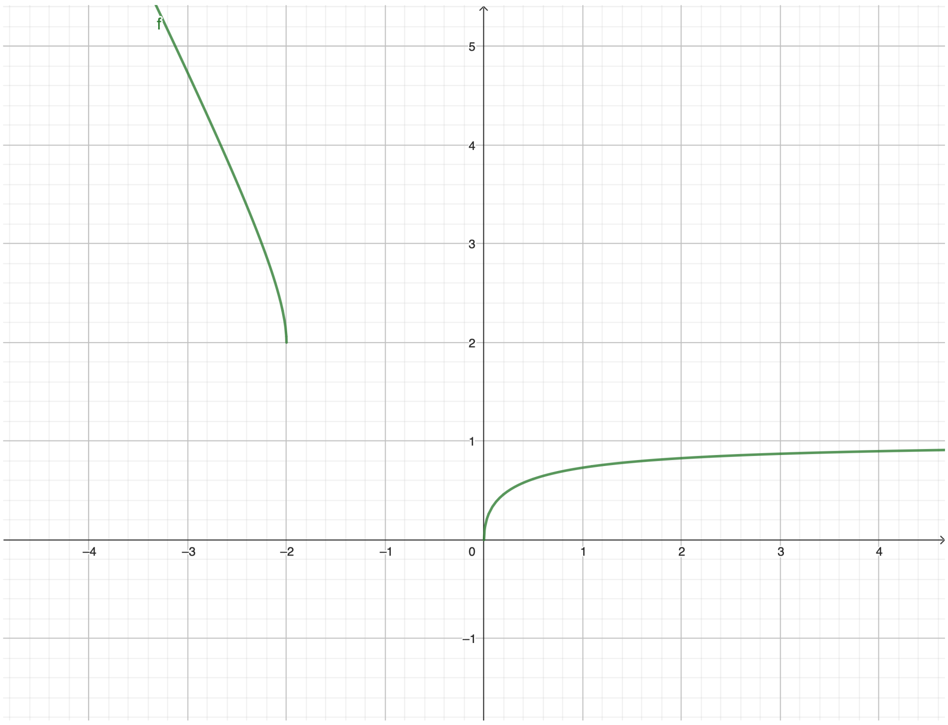

Figure 3: Graph di f

In particular, f is not differentiable at the points $x = -2$ and $x = 0$. For every $x \in (-\infty, -2) \cup (0, +\infty)$ we have

$$f'(x) = 0 \Leftrightarrow x + 1 = \sqrt{x^2 + 2x}$$

and there are no solutions to this equation because$(x + 1)^2 = x^2 + 2x + 1 \neq x^2 + 2x = (\sqrt{x^2 + 2x})^2$. Moreover $f'(x) < 0$ for every $x \in (-\infty, -2)$ and $f'(x) > 0$ for every $x \in (0, +\infty)$, hence f is decreasing in $(-\infty, -2]$ and increasing in $[0, +\infty)$; moreover -2 and 0 are relative minimum points, 0 is an absolute minimum point and $f(0) = 0$ is the infimum (actually minimum) di f. There are no relative maximum points and the supremum is $+\infty$.

(d) plot a qualitative graph of f.

See picture 3.

Exercise 2 (score 7) Consider the complex polynomial equation:

$$z^3 + \alpha z^2 + iz = -\alpha i \qquad (\alpha \in \mathbb{R})$$

(a) Determine the value of the parameter α such that this equation has $z_0 := 3$ as a solution;

By imposing that the equation holds true for $z = 3$ we get $27 + 9\alpha + 3i = -\alpha i$ hence $\alpha = -\frac{27+3i}{9+i} = -3\frac{9+i}{9+i} = -3$.

(b) If α is as in pont a), find the remaining solutions of the equation.

We have $z^3 - 3z^2 + iz - 3i = (z - 3)(z^2 + i)$ hence the remaining solutions are the square roots of $-i$: to compute them let us observe that $|-i| = 1$ and $\text{Arg}(-i) = -\frac{\pi}{2}$. As a consequence the two roots z_1 and z_2 have modulus 1 and argument $-\frac{\pi}{4}$ and $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$, respectivaly, so that

$$z_1 = e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

$$z_2 = e^{i\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

As an alternative method, we might consider the factorization $z^3 + \alpha z^2 + iz + \alpha i = (z^2 + i)(z + \alpha)$ and conclude in a obvious way.

Exercise 3 (score 8) (a) Compute the limit

$$\lim_{x\to 0^+} [1 - \sin(x)]^{\frac{1}{x}}.$$

Since the exponential function is continuous, we have

$$\lim_{x\to 0^+} [1 - \sin(x)]^{\frac{1}{x}} = e^{\lim\limits_{x\to 0^+} \frac{\log(1-\sin(x))}{x}} = \frac{1}{e},$$

because

$$\lim_{x\to 0^+} \frac{\log(1-\sin(x))}{x} = \lim_{x\to 0^+} \frac{\log(1-(x+o(x))}{x} = \lim_{x\to 0^+} \frac{-(x+o(x)) + o\big(-x-o(x)\big)}{x} =$$

$$\lim_{x\to 0^+} \frac{-x + o\big(x\big)}{x} = -1$$

(b) Study the character of the seris

$$\sum_{n=1}^{+\infty} \left[1 - \sin\left(\frac{1}{n}\right)\right]^{n^2}.$$

We begin by observing that the terms of this series are positive. Therefore we can apply the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left[1 - \sin\left(\frac{1}{n}\right)\right]^{n^2}} = \lim_{n\to\infty} \left[1 - \sin\left(\frac{1}{n}\right)\right]^{n} = \frac{1}{e}$$

because, by the change of variable $x = \frac{1}{n}$ and by point (a), we get

$$\lim_{n\to\infty} \left[1 - \sin\left(\frac{1}{n}\right)\right]^{n} = \lim_{x\to 0^+} [1 - \sin(x)]^{\frac{1}{x}} = \frac{1}{e} < 1,$$

so the series is convergent.

Exercise 4 (score 8) Consider the family of functions

$$f_\alpha(x) = (x-1)\arctan(x^\alpha)$$

(a) Compute

$$\int f_1(x)\,dx$$

$$\int f_1(x)\,dx = \int \big[(x-1)\arctan x\big]\,dx = \frac{(x-1)^2}{2}\arctan x - \int \left[\frac{(x-1)^2}{2(1+x^2)}\right] dx + c \qquad c \in \mathbb{R}$$

Now

$$\frac{(x-1)^2}{2(1+x^2)} = \frac{x^2+1}{2(x^2+1)} - \frac{1}{2}\frac{4x}{2(x^2+1)}$$

so that

$$\int f_1(x)\,dx = \frac{(x-1)^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\log(2(x^2+1)) + c \qquad c \in \mathbb{R}$$

(b)Determine for which values of the parameter $\alpha \in \mathbb{R}$, the integral

$$\int_1^{+\infty} f_\alpha(x)\,dx$$

is convergent.

There is no problem at the extreme 1, for f_α is continuous at every $x \geq 1$. Hence we have to compute

$$\lim_{k\to+\infty}\int_1^k f_\alpha(x)\,dx = \lim_{k\to+\infty}\int_1^k \big((x-1)\arctan(x^\alpha)\big)\,dx$$

If $\alpha \geq 0$, one has $f_\alpha \sim (x-1)$ for $x \to +\infty$, so, by the asymptotic comparison test, the integral is not convergent.

If instead $\alpha < 0$, by $\arctan y = y + o(y)$ one has $f_\alpha \sim \frac{1}{x^{-1-\alpha}}$, so by the asymptotic comparison test, the integral is convergent if and only if $-1-\alpha > 1$, i.and., if and only if $\alpha < -2$

Some Taylor expansions:

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}),$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \qquad \forall n \geq 0$$

CALCULUS 1

Information Engineering

23.01.2023

TEMA 4

Exercise 1 (score 9) Consider the function

$$f(x) = x - \sqrt{x^2 - 2x}$$

(a) find the domain of f ;

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : x^2 - 2x \geq 0\} = (-\infty, 0] \cup [2, +\infty).$$

(b) compute the limits at significative points and asymptotes

f is continuous for every $x \in \mathrm{Dom}(f)$ hence

$$\lim_{x\to 0^-} f(x) = f(0) = 0, \qquad \lim_{x\to 2^+} f(x) = f(2) = 2;$$

moreover

$$\lim_{x\to -\infty} f(x) = -\infty$$

and

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} f(x)\frac{x + \sqrt{x^2 - 2x}}{x + \sqrt{x^2 - 2x}} = \lim_{x\to +\infty} \frac{x}{x + \sqrt{x^2 - 2x}} = 1,$$

in particular, $y = 1$ is horizontal asymptote for $x \to +\infty$. Let us compute the asymptote for $x \to -\infty$:

$$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} 1 + \sqrt{1 - \frac{2}{x}} = 2$$

and

$$\lim_{x\to -\infty} f(x) - 2x = \lim_{x\to -\infty} -\left(x + \sqrt{x^2 - 2x}\right)\frac{x - \sqrt{x^2 - 2x}}{x - \sqrt{x^2 - 2x}} = \lim_{x\to -\infty} -\frac{2x}{x - \sqrt{x^2 - x}} = -1.$$

Hence $y = 2x - 1$ is asymptote for $x \to -\infty$.

(c) Study the differentiability of f, compute the derivative and possible limits of the derivative; discuss the monotonicity of f, determine if f is upper [lower] bounded, and in the positive case find the supremum [infimum], and relative and absolute minima and maxima (if they exist) ;

f is differentiable in $(-\infty, 0) \cup (2, +\infty)$ because it is a composition and sum of differentiable functions; in particular, $x^2 - 2x > 0$ in $(-\infty, 0) \cup (2, +\infty)$ and $\sqrt{y}$ is differentiable for $y > 0$. At these points the derivative is

$$f'(x) = 1 - \frac{x - 1}{\sqrt{x^2 - 2x}}.$$

As for the limits of the derivative, one has:

$$\lim_{x\to 0^-} f'(x) = +\infty, \quad \lim_{x\to 2^+} f'(x) = -\infty,$$

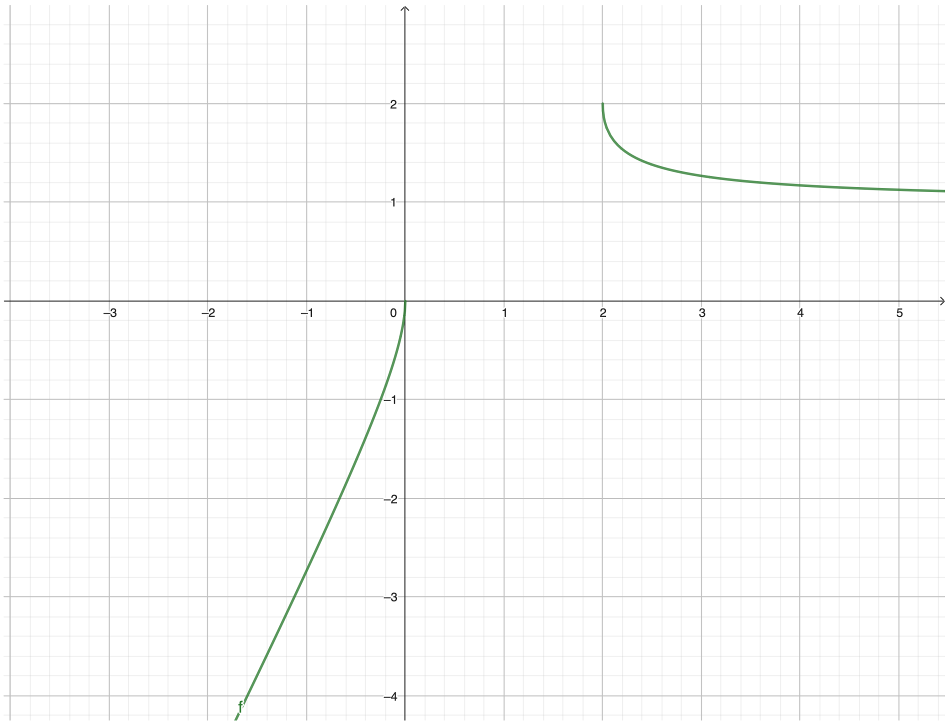

Figure 4: Graph di f

In particular, f is not differentiable nei points $x = 0$ and $x = 2$. For every $x \in (-\infty, 0) \cup (2, +\infty)$ we have

$$f'(x) = 0 \Leftrightarrow x - 1 = 2\sqrt{x^2 - 2x}$$

and there are no solutions to this equation because$(x-1)^2 = x^2 - 2x + 1 \neq x^2 - 2x = (\sqrt{x^2 - 2x})^2$. Moreover $f'(x) > 0$ for every $x \in (-\infty, 0)$ and $f'(x) < 0$ for every $x \in (2, +\infty)$, hence f is increasing in $(-\infty, 0]$ and decreasing in $[2, +\infty)$; moreover 0 and 2 are relative maximum points, 2 is an absolute maximum point, and $f(2) = 2$ is the supremum (maximum) di f. There are no relative minimum points and the function is not lower bounded;

(d) plot a qualitative graph of f.

See picture 4.

Exercise 2 (score 7) Consider the complex polynomial equation:

$$z^3 + \alpha z^2 + iz = -\alpha i \qquad (\alpha \in \mathbb{R})$$

(a) Determine the value of the parameter α such that this equation has $z_0 := -3$ as a solution;

By imposing that the equation holds true for $z = -3$ we get $-27 + 9\alpha - 3i = -\alpha i$ hence $\alpha = \frac{27+3i}{9+i} = 3\frac{9+i}{9+i} = 3$.

(b) If α is as in pont a), find the remaining solutions of the equation.

We have $z^3 + 3z^2 + iz + 3i = (z+3)(z^2 + i)$ hence le altre soluzioni sono le due radici quadrate di $-i$: for calcolarle osserviamo che $|-i| = 1$ and $\mathrm{Arg}(-i) = -\frac{\pi}{2}$ and of conseguenza le due radici z_1 and z_2 hanno modulo 1 and argument $-\frac{\pi}{4}$ and $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ respectively, that is

$$z_1 = e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

$$z_2 = e^{i\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

In alternative, we might have observed that $z^3 + \alpha z^2 + iz + \alpha i = (z^2 + i)(z + \alpha)$, hence the solutions are $-\alpha, z_1, z_2$: the roots z_1 and z_2 are computed as in (b) and for (a)(to be satisfied it is necessary and sufficient that $-\alpha = -3$.

Exercise 3 (score 8) (a) Compute the limit

$$\lim_{x\to 0^+} [1 - \tan(x)]^{\frac{1}{x}}.$$

Since the exponential function is continuous, we have

$$\lim_{x\to 0^+} [1 - \tan(x)]^{\frac{1}{x}} = e^{\lim\limits_{x\to 0^+} \frac{\log(1-\tan(x))}{x}} = \frac{1}{e},$$

because

$$\lim_{x\to 0^+} \frac{\log(1 - \tan(x))}{x} = \lim_{x\to 0^+} \frac{\log(1 - (x + o(x))}{x} = \lim_{x\to 0^+} \frac{-(x + o(x)) + o\big(-x - o(x)\big)}{x} =$$

$$\lim_{x\to 0^+} \frac{-x + o\big(x\big)}{x} = -1$$

(b) Study the character of the seris

$$\sum_{n=1}^{+\infty} \left[1 - \tan\left(\frac{1}{n}\right)\right]^{n^2}.$$

We begin by observing that the terms of this series are positive for every $n \geq 2$. Therefore we can apply the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left[1 - \tan\left(\frac{1}{n}\right)\right]^{n^2}} = \lim_{n\to\infty} \left[1 - \tan\left(\frac{1}{n}\right)\right]^n = \frac{1}{e}$$

because, by the change of variable $x = \frac{1}{n}$ and by point (a), we get

$$\lim_{n\to\infty} \left[1 - \tan\left(\frac{1}{n}\right)\right]^n = \lim_{x\to 0^+} [1 - \tan(x)]^{\frac{1}{x}} = \frac{1}{e} < 1,$$

so the series is convergent.

Exercise 4 (score 8) Consider the family of functions

$$f_\alpha(x) = (x + 2)\arctan(x^\alpha)$$

(a) Compute

$$\int f_1(x)\, dx$$

$$\int f_1(x)\, dx = \int \big[(x + 2)\arctan x\big] dx = \frac{(x + 2)^2}{2} \arctan x - \int \left[\frac{(x + 2)^2}{2(1 + x^2)}\right] dx + c \qquad c \in \mathbb{R}$$

Now

$$\frac{(x + 2)^2}{2(1 + x^2)} = \frac{x^2 + 1}{2(x^2 + 1)} + \frac{3}{2(x^2 + 1)} + \frac{4x}{2(x^2 + 1)}$$

so that

$$\int f_1(x)\,dx = \frac{(x+2)^2-3}{2}\arctan x - \frac{1}{2}x - \log(2(x^2+1)) + c \qquad c \in \mathbb{R}$$

(b)Determine for which values of the parameter $\alpha \in \mathbb{R}$, the integral

$$\int_1^{+\infty} f_\alpha(x)\,dx$$

is convergent.

There is no problem at the extreme 1, for f_α is continuous at every $x \geq 1$. Hence we have to compute

$$\lim_{k\to+\infty}\int_1^k f_\alpha(x)\,dx = \lim_{k\to+\infty}\int_1^k \big((x+2)\arctan(x^\alpha)\big)\,dx$$

If $\alpha \geq 0$, one has $f_\alpha \sim (x+2)$ for $x \to +\infty$, so, by the asymptotic comparison test, the integral is not convergent.

If instead $\alpha < 0$, by $\arctan y = y + o(y)$ one has $f_\alpha \sim \frac{1}{x^{-1-\alpha}}$, so by the asymptotic comparison test, the integral is convergent if and only if $-1-\alpha > 1$, i.and., if and only if $\alpha < -2$

Some Taylor expansions:

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6),$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2}) \qquad \forall n \geq 0$$