Answer the following questions, providing a motivation for each answer.
(In case of negative answer the "motivation" is a counterexample)

1. Is it true that a continuous function defined on the interval $]1,3]$ has an absolute maximum and an absolute minimum?

   NO. A counterexample is the function $f:]1,3] \to \mathbb{R}$,

   $$f(x) = \begin{cases} -x^2 & \forall x \in]1,2[\\ -6+x & \forall x \in [2,3] \end{cases}$$

   f has a absolute minimum at $x=2$ but it hasn't an absolute maximum.

2. Is it true that a continuous function defined on the interval $]1,3]$ has a relative maximum and a relative minimum?

   NO. A counterexample is the function $f:]1,3] \to \mathbb{R}$,

   $$f(x) = \begin{cases} -x^2 & \forall x \in]1,2[\\ -2-x & \forall x \in [2,3] \end{cases}$$

   f has an absolute (hence relative) minimum at $x=3$ but it hasn't an absolute maximum.

3. Consider a function $f:]-1,1[\to \mathbb{R}$, and assume that it is differentiable on $]-1,1[\setminus\{0\}$ and that the limit $\lim\limits_{x\to 0} f'(x)$ does NOT exist. Can we conclude that the function f is not differentiable at $x=0$?

   NO. The function

   $$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \forall x \neq 0 \\ 0 & \text{for } x=0 \end{cases}$$

   is differentiable at any point: indeed, if $x \neq 0$ $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$, while for $x=0$ one can compute the derivative as limit of the differential quotient (do it by exercise), so obtaining that $f'(0)=0$.

   However the limit

   $$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

   does not exist.

4. Suppose that a function $f:[0,1]\to\mathbb{R}$ is differentiable at all points of $]0,1[$ and that f is continuous at $x=0$ and $x=1$. Can we conclude that f is integrable?

   YES. Indeed f is differentiable, hence continuous on $]0,1[$, moreover it is continuous at $x=0$ and $x=1$. Therefore it is continuous on the closed and bounded interval $[0,1]$, hence it is integrable on $[0,1]$.

5. Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series which does not converge absolutely. Can we conclude that

   $$\lim_{n\to\infty} (a_n)^4 = 0?$$

   YES. A necessary condition for the series to be convergent is $\lim_{n\to\infty} a_n = 0$. Hence one also has $\lim_{n\to\infty} |a_n| = 0$. In particular there is $\bar{n}$ such that $|a_n| < 1$ for all $n > \bar{n}$. Therefore, for all $n > \bar{n}$ one has $0 \le a_n^4 = |a_n|^4 \le |a_n|$, and by the two policemen theorem one concludes that $\lim_{n\to\infty} a_n^4 = 0$

6. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Is it true that the series $\sum_{n=1}^{\infty} b_n$, with $b_n = -a_n \ \forall n \in \mathbb{N}$, is a convergent series as well?

   YES. Since $\sum_{n=1}^{\infty} a_n$ is convergent one has $\lim_{k\to\infty} S_k^a = S \in \mathbb{R}$, where, for every k, $S_k^a := a_1 + \cdots + a_k$. Setting $S_k^b := b_1 + \cdots + b_k = -a_1 + \cdots - a_k = -S_k^a$, one gets $\lim_{k\to\infty} S_k^b = -S$, i.e. the series $\sum_{n=1}^{\infty} b_n$ converges (and its sum is $-S$).

7. Consider a complex polynomial $P(z) = b_5 z^5 + b_4 z^4 + b_3 z^3 + b_2 z^2 + b_1 Z + b_0 \quad (b_0, \ldots, b_5 \in \mathbb{C})$. Is it true that the equation $P(z) = 0$ has at least one real solution?

   NO. A simple counterexample is

   $$P(z) := (z+i)^5 = z^5 + 5iz^4 - 10z^3 - 10iz^2 + 5iz + i$$

   and the only solution of $P(z) = 0$ is $-i$.

8. Same as in the previous question but with $b_0, \ldots, b_5 \in \mathbb{R}$

   YES. By the Fundamental Theorem of Algebra we know that there are 5 solutions (when counted with their multiplicity). Moreover, since the coefficients of $P(z)$ are real, for every solution $z = x+iy$, also its conjugate $\bar{z} = x - iy$ is a solution. Therefore, there must be at least one solution $\hat{z}$ which coincides with its conjugate. This is possible if and only if $\bar{z}$ is a real number.

9. Consider a function $f : [0, +\infty[\to \mathbb{R}$ and assume that the series $\sum_{n=1}^{\infty} a_n$, where $a_n := f(n) \ \forall n \in \mathbb{N}$, converges. Can we conclude that $\lim_{x\to\infty} f(x) = 0$?

   NO. A counterexample is

   $$f(x) := \begin{cases} \frac{1}{x^2} & \forall x \in \mathbb{N} \\ 1 & \forall x \in [0, +\infty[\setminus \mathbb{N} \end{cases}$$

   The limit $\lim_{x\to\infty} f(x)$ does not exist.