Geometria 1 - mod. A - Lezione 39

Note Title

Esercizio di ieri: $x^2+x+1 = m_A(x) \in \mathbb{R}$

Sia $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ con $p_\varphi(x) = x^2+x+1$, $\varphi^3 = id$ $\varphi \neq id$

Se lavoro su $\mathbb{C}^2 \xrightarrow{\varphi} \mathbb{C}^2$ sia $\alpha_{\mathcal{E}\mathcal{E}}(\varphi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \in M_2(\mathbb{R})$

$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ $\quad p_\varphi(x) \in \mathbb{C}[x]$ rimane uguale.

$$(x^2+x+1) = (x-\alpha)(x-\bar{\alpha}) \qquad \alpha = -\frac{1}{2}+\frac{\sqrt{3}}{2}i \neq \bar{\alpha} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

Deduco che φ è diagonalizzabile su $\mathbb{C}$

$$\mathbb{C}^2 = V_\alpha \oplus V_{\bar{\alpha}}$$
$$V_\alpha = \langle v \rangle, \quad V_{\bar\alpha} = \langle \bar{v} \rangle \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} \in \mathbb{C}^2$$

$$\overline{\varphi(v)} = \overline{Av} = \bar{A}\bar{v} = A\bar{v}$$
$$\overline{\varphi(v)} = \overline{(\alpha v)} = \bar{\alpha}\bar{v}$$

$$v = v_r + i v_c \qquad v_r, v_c \in \mathbb{R}^2$$
$$v = \begin{pmatrix} \mathrm{Re}\, v_1 + i\, \mathrm{Im}\, v_1 \\ \mathrm{Re}\, v_2 + i\, \mathrm{Im}\, v_2 \end{pmatrix} \qquad v_r = \begin{pmatrix} \mathrm{Re}\, v_1 \\ \mathrm{Re}\, v_2 \end{pmatrix} \quad v_c = \begin{pmatrix} \mathrm{Im}\, v_1 \\ \mathrm{Im}\, v_2 \end{pmatrix}$$

$$v + \bar{v} = v_r + \cancel{i v_c} + v_r - \cancel{i v_c} = 2 v_r \qquad v_r = \frac{v+\bar{v}}{2}$$
$$v - \bar{v} = v_r + i v_c - v_r + i v_c = 2i v_c \qquad v_c = \frac{v-\bar{v}}{2i}$$
$$\alpha = \underbrace{-\frac{1}{2}}_{\alpha_1} + \underbrace{\frac{\sqrt{3}}{2}}_{\alpha_2} i \qquad \alpha_1 = \frac{\alpha+\bar{\alpha}}{2} \qquad \alpha_2 = \frac{\alpha-\bar{\alpha}}{2i}$$

$$\varphi(v_r) = \varphi\left(\frac{v+\bar{v}}{2}\right) = \frac{1}{2}[\varphi(v)+\varphi(\bar{v})] = \frac{1}{2}[\alpha v + \bar{\alpha}\bar{v}] = \text{"}\mathrm{Re}\,\alpha v\text{"}$$
$$= \frac{1}{2}\Big(\underbrace{\alpha_1 v_r - \alpha_2 v_c}_{w_1} + i\underbrace{(\alpha_1 v_c + \alpha_2 v_r)}_{w_2} + w_1 - \cancel{i w_2}\Big)$$
$$= \alpha_1 v_r - \alpha_2 v_c$$

$$\varphi(v_c) = \varphi\left(\frac{v-\bar{v}}{2i}\right) = \text{"}\mathrm{Im}\,\alpha v\text{"} = \alpha_1 v_c + \alpha_2 v_r$$

$v_r, v_c \in \mathbb{R}^2$ sono l. ind.

La matrice di φ nella base $\{v_R, v_C\}$ è $\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix} = B$

$$\begin{pmatrix} -1/2 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

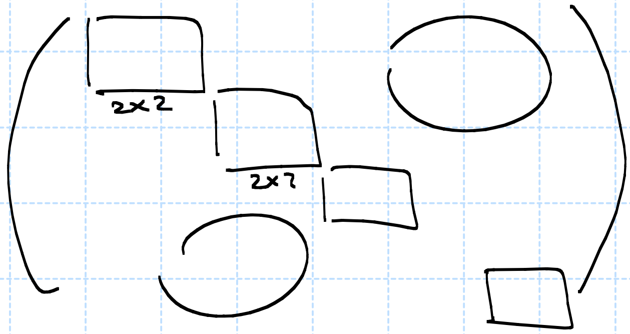
Vale in generale anche per polinomi irriducibili in $\mathbb{R}[x]$ del tipo $\underbrace{x^2+bx+c}_{\substack{\in \mathbb{R}[x] \\ \parallel \\ (x-\alpha)(x-\bar{\alpha})}}$

$$\alpha_1 + i\alpha_2 = \alpha \in \mathbb{C}$$

esiste base t.c. la matrice di φ sia del tipo B.

In generale $\varphi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ $\quad p_\varphi(x) = \prod_{i=1}^{n} (\underbrace{x^2 + b_i x + c_i}_{\substack{\text{irriducibili} \\ \text{coprimi}}})$

posso trovare una base di $\mathbb{R}^{2n}$ t.c. la matrice di φ



con blocchi diagonali 2×2 del tipo B con i coeff. che dipendono dalle radici complesse dei polinomi $x^2 + b_i x + c_i$.

Esercizio: Jordanizzare

$$\begin{pmatrix} 0-x & 0 & 1 & -1 & 0 \\ 0 & 1-x & 0 & 0 & -1 \\ 1 & 0 & 0-x & 0 & 0 \\ 1 & 0 & 0 & 0-x & 0 \\ 0 & 1 & 0 & 0 & -1-x \end{pmatrix} = A$$

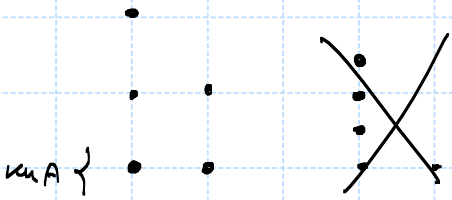
$$-p_A(x) \overset{\text{sviluppo risp. 5ª colonna}}{=} \begin{vmatrix} -x & 0 & 1 & -1 \\ 1 & 0 & -x & 0 \\ 1 & 0 & 0 & -x \\ 0 & 1 & 0 & 0 \end{vmatrix} - (1+x) \begin{vmatrix} -x & 0 & 1 & -1 \\ 0 & 1-x & 0 & 0 \\ 1 & 0 & -x & 0 \\ 1 & 0 & 0 & -x \end{vmatrix}$$

$$= \begin{vmatrix} -x & 1 & -1 \\ 1 & -x & 0 \\ 1 & 0 & -x \end{vmatrix} - \underbrace{(1+x)(1-x)} \begin{vmatrix} -x & 1 & -1 \\ 1 & -x & 0 \\ 1 & 0 & -x \end{vmatrix}$$

$$= \begin{vmatrix} -x & 1 & -1 \\ 1 & -x & 0 \\ 1 & 0 & -x \end{vmatrix} [\cancel{1} - (\cancel{1} - x^2)] = x^2(-x^3 \cancel{-x} \cancel{+x}) = -x^5$$

$p_A(x) = x^5$ Dunque A è nilpotente.

$$\operatorname{Ker} A = \operatorname{Ker}\begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 0\\0\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\1 \end{pmatrix} \right\rangle$$



$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \operatorname{Ker} A^2 = \left\langle \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} \right\rangle$$

$x_3 + x_4 = 0$

$e_3 \notin \operatorname{Ker} A^2$

opp. e_4 opp. $e_1 + e_3$

$$A^3 = 0 \implies \operatorname{Ker} A^3 = \mathbb{R}^5$$

$$\begin{tikzcd} v_3 \arrow[d, bend right, "\varphi"'] \\ v_2 \arrow[d, bend right, "\varphi"'] \\ v_1 \end{tikzcd} \qquad \begin{tikzcd} v_5 \arrow[d, bend left, "\varphi"] \\ v_4 \end{tikzcd}$$

$\underline{\underline{v_3}} \in \operatorname{Ker} A^3 \setminus \operatorname{Ker} A^2$, con $\operatorname{Ker} A^3 = \mathbb{R}^5$

scelta

Sia $v_3 = e_3$ $\quad v_2 = \varphi(e_3) = e_1 \in \operatorname{Ker} A^2$

$$v_1 = \varphi(v_2) = \varphi^2(e_3) = \begin{pmatrix} 0\\0\\1\\1\\0 \end{pmatrix} \in \operatorname{Ker} A$$

$$\underline{\underline{v_5}} \in \operatorname{Ker} A^2 \setminus \langle \operatorname{Ker} A, v_2 \rangle = \operatorname{Ker} A^2 \setminus \left\langle \begin{pmatrix} 0\\0\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix} \right\rangle$$

scelta

ad esempio $e_5 = v_5$

$$v_4 = \varphi(e_5) = \begin{pmatrix} 0\\-1\\0\\0\\-1 \end{pmatrix} \in \operatorname{Ker} A$$

NB è l. ind. da v_1

Rispetto alla base $\begin{pmatrix} 0\\0\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix}, e_3, -e_2-e_5, e_5$ la matrice di φ

è nella forma

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix} = H^{-1} A H$$

$J_3 \qquad J_2$

$$H^{-1} A H = \alpha_{EV}(id) \cdot \alpha_{EE}(\varphi) \cdot \underbrace{\alpha_{VE}(id)}_{H}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} = H$$

Esercizio: sia $f: K^4 \longrightarrow K^3$ lineare

$$f(e_1)=\begin{pmatrix}1\\-1\\2\end{pmatrix} \quad f(e_2)=\begin{pmatrix}0\\-1\\1\end{pmatrix} \quad f(e_3)=\begin{pmatrix}1\\0\\-1\end{pmatrix} \quad f(e_4)=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\alpha_{\mathcal{E}\mathcal{E}}(f)=\begin{pmatrix}1&0&1&1\\-1&-1&0&0\\2&1&-1&1\end{pmatrix}=A \qquad 4=\underbrace{\dim \ker f}_{\text{nullità}}+\underbrace{\dim \operatorname{Im} f}_{\substack{\text{rango di } f\\ \parallel \\ 3}}$$

$\ker(f)=\ker A$

$$\left(\begin{array}{cccc|c}1&0&1&1&0\\-1&-1&0&0&0\\2&1&-1&1&0\end{array}\right)\underset{\mathrm{III}-2\mathrm{I}}{\overset{\mathrm{I}+\mathrm{II}}{\sim}}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&+1&-1&-1&0\\0&\cancel{1}\,0&\cancel{-3}\,-2&\cancel{-1}\,0&0\end{array}\right)$$

$-\mathrm{II}$

$\mathrm{III}-\mathrm{II}$

$$\sim\left(\begin{array}{cccc|c}\boxed{1}&0&\cancel{1}\,0&1&0\\0&\boxed{1}&\cancel{-1}\,0&-1&0\\0&0&\boxed{1}&0&0\end{array}\right)$$

rango $A=3 \Longrightarrow \operatorname{Im} f = K^3 \Longrightarrow f$ suriett.

$\dim \ker A = 1 \qquad \ker A = \left\langle \begin{pmatrix}1\\-1\\0\\-1\end{pmatrix} \right\rangle$

- Cercare, se esiste, una base $\mathcal{B}$ di K^4 t.c. $\alpha_{\mathcal{B}\mathcal{E}}(f)=\left(\underbrace{\begin{matrix}1&&0\\&\ddots&\\0&&1\end{matrix}}_{\mathbb{1}_3}\begin{matrix}0\\0\\0\end{matrix}\right)$

$\mathcal{B}=\{v_1, v_2, v_3, v_4\}$

$f(v_1)=e_1, \quad f(v_2)=e_2, \quad f(v_3)=e_3, \quad f(v_4)=0$

v_1, v_2, v_3 esistono perché f è suriettiva $\qquad$ Ad es $v_4=\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$

In generale dato $b=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\in K^3$, $\quad f^{-1}(b)\subseteq K^4$

$\uparrow$ devo risolvere $(Ax=b)$

$$\left(\begin{array}{cccc|c}1&0&1&1&b_1\\-1&-1&0&0&b_2\\2&1&-1&1&b_3\end{array}\right) \underset{\text{con Gauss}}{\text{risolvo}} \left(\begin{array}{cccc|c}1&0&0&1&b_1-\frac{b_1-b_2-b_3}{2}\\0&1&0&-1&-b_2-b_1+\frac{b_1-b_2-b_3}{2}\\0&0&1&0&\frac{b_1-b_2-b_3}{2}\end{array}\right)$$

$$\begin{pmatrix}b_1-\dots\\-b_2-b_1+\frac{b_1-b_2-b_3}{2}\\\frac{b_1-b_2-b_3}{2}\\0\end{pmatrix}\in f^{-1}(b)$$

Scelgo come $v_1\in f^{-1}(e_1)$ $\quad b_1=1,\ b_2=0,\ b_3=0$ $\quad v_1=\begin{pmatrix}1/2\\-1/2\\1/2\\0\end{pmatrix}$ $\quad$ controllare!

$v_2\in f^{-1}(e_2)$ $\quad b_1=0,\ b_2=1,\ b_3=0$ $\quad v_2=\begin{pmatrix}1/2\\-3/2\\-1/2\\0\end{pmatrix}$

$v_3\in f^{-1}(e_3)$ $\quad b_1=b_2=0,\ b_3=1$ $\quad v_3=\begin{pmatrix}1/2\\-1/2\\-1/2\\0\end{pmatrix}$

Ad es. posso prendere $\mathcal{B}=\left\{v_1, v_2, v_3, \begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}\right\}$ con v_i

$$f : K^4 \longrightarrow K^3 = \mathrm{Im} f = \underset{\substack{\| \\ \langle f(e_1)\rangle \\ \| \\ \left\langle \begin{pmatrix}1\\-1\\2\end{pmatrix} \right\rangle}}{U} \oplus \underset{\substack{\| \\ \langle f(e_2), f(e_3)\rangle \\ \| \\ \left\langle \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\rangle}}{W}$$

$u_1 = \begin{pmatrix}1\\-1\\2\end{pmatrix}$ $\qquad \begin{cases} x_1 + x_2 = 0 \\ 2x_1 - x_3 = 0 \end{cases}$

$u_2 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$, $u_3 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\qquad$ eq $x_1 + x_2 + x_3 = 0$

Considero $\pi_U^W : K^3 \longrightarrow K^3$

$$\alpha_{\mathcal{E}\mathcal{E}}(\pi_U^W) = ?$$

Se $\mathcal{U} = \{u_1, u_2, u_3\}$ $\qquad \alpha_{\mathcal{U}\mathcal{U}}(\pi_U^W) = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} =: A'$

$$B' = \alpha_{\mathcal{E}\mathcal{E}}(\pi_U^W) = P^{-1} A' P \qquad P = \alpha_{\mathcal{E}\mathcal{U}}(id)$$

$$P^{-1} = \alpha_{\mathcal{U}\mathcal{E}}(id) = \begin{pmatrix}1&0&1\\-1&-1&0\\2&1&-1\end{pmatrix} \quad \text{Lavoro con Gauss e Trovo} \longrightarrow = \begin{pmatrix}1/2&1/2&1/2\\-1/2&-3/2&-1/2\\1/2&-1/2&-1/2\end{pmatrix}$$

perché coincide con le prime coordinate di v_1, v_2, v_3?

Altro metodo per calcolare B'.

$$\pi_U^W(e_1) = \pi_U^W\left(\alpha_1 \underbrace{\begin{pmatrix}1\\-1\\2\end{pmatrix}}_{u_1} + \alpha_2 \underbrace{\begin{pmatrix}0\\-1\\1\end{pmatrix}}_{u_2} + \alpha_3 \underbrace{\begin{pmatrix}1\\0\\-1\end{pmatrix}}_{u_3}\right) = \alpha_1 \begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&0&1&b_1\\-1&-1&0&b_2\\2&1&-1&b_3\end{array}\right) \leadsto \left(\begin{array}{ccc|c}1&0&0&b_1 - \frac{b_1-b_2-b_3}{2}\\0&1&0&-b_2-b_1+\frac{b_1-b_2-b_3}{2}\\0&0&1&\frac{b_1-b_2-b_3}{2}\end{array}\right) \begin{array}{l}\to \beta_1\\ \to \beta_2 \\ \to \beta_3\end{array}$$

Soluzioni sono $\begin{pmatrix}\beta_1\\\beta_2\\\beta_3\end{pmatrix}$ t.c. $\beta_1\begin{pmatrix}1\\-1\\2\end{pmatrix} + \beta_2\begin{pmatrix}0\\-1\\1\end{pmatrix} + \beta_3\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

coordinate di e_1 nella base $\mathcal{U}$ sono $\begin{pmatrix}1/2\\-1/2\\1/2\end{pmatrix}$; $e_1 = \frac{1}{2}u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_3$

e_2 - - - - $\begin{pmatrix}1/2\\-3/2\\-1/2\end{pmatrix}$

e_3 - - - - $\begin{pmatrix}1/2\\-1/2\\-1/2\end{pmatrix}$

$$\pi_W^W(e_1) = \frac{1}{2}u_1 = \pi_U^W(e_2) = \pi_U^W(e_3) = \begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}$$

$$\alpha_{\mathcal{E}\mathcal{E}}(\pi_U^W) = \begin{pmatrix}1/2&1/2&1/2\\-1/2&-1/2&-1/2\\1&1&1\end{pmatrix}$$

$$\begin{tikzcd} K^4 \arrow[r, "f"] \arrow[rr, bend right, "\pi_U^W \circ f"'] & K^3 \arrow[r, "\pi_U^W"] & K^3 \end{tikzcd}$$

$$\alpha_{\mathcal{E}\mathcal{E}}(\pi_U^W \circ f) = \alpha_{\mathcal{E}\mathcal{E}}(\pi_U^W)\, \alpha_{\mathcal{E}\mathcal{E}}(f) \qquad (*)$$

Esercizio: calcolarlo!

- Se invece di considerare π_U^W avessi considerato π_W^U avrei cercato

$$id = \pi_U^W + \pi_W^U \qquad \alpha_{\mathcal{E}\mathcal{E}}(\pi_W^U) = \mathbb{1} - \alpha(\pi_U^W)$$

- Scrivere la matrice $\alpha_{\mathcal{E}\mathcal{E}}(\pi \circ f)$

  C'è un modo più facile rispetto ad applicare $(*)$?

- Domanda:

$$\begin{tikzcd} K^4 \arrow[r, "f"] \arrow[d, "\pi_Z^{\ker f}"'] & K^3 \arrow[d, "\pi_U^W"] \\ K^4 \arrow[r, dashed, "\exists ? h"] & K^3 \end{tikzcd}$$

$K^4 = \ker f \oplus Z$ ← sceglielo!

esiste h t.c. $\pi_U^W \circ f = h \circ \pi_Z^{\ker}$?

- <u>Esercizio per casa!</u>

$$A = \begin{pmatrix} 2 & 4 & -k & 0 \\ 4 & k & -3 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & -k & 2 \end{pmatrix} \in M_4(\mathbb{R})$$

- È possibile diagonalizzare A su $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$?
- È possibile scriverla in forma di Jordan su $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$?

Discutere in funzione di $k \in \mathbb{R}$.

Considerare $k = 4$ (risp. $k = 2$) e Jordanizzare.