V, $\varphi: V \to V$ lineare, $P_\varphi(x) = (x-c_1)^{m_1} \dots (x-c_r)^{m_r}$

$m_i = ma(c_i)$ $\quad (m_1 + \dots + m_r = n = \dim V)$

- $(x-c_1)^{m_1}, \dots, (x-c_r)^{m_r}$ sono a 2 a 2 coprimi
- $P_\varphi(x) \in \operatorname{Ker}(ev_\varphi)$

↳ Lemma di decomposizione:

$$V = W_1 \oplus \dots \oplus W_r, \quad W_i = \operatorname{Ker}(\varphi - c_i \cdot id_V)^{m_i}$$

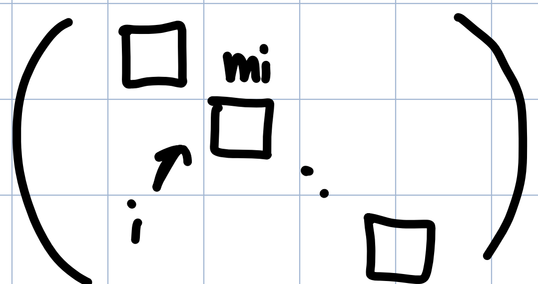
W_i = autospazio generalizzato relativo a c_i

$\varphi(W_i) \subseteq W_i$

Autovett. gen. rel. a c_i: $w \neq 0$ t.c. $(\varphi - c_i)^{m_i}(w) = 0$

PERIODO r: $(\varphi - c_i)^r(w) = 0$

$(\varphi - c_i)^{r-1}(w) \neq 0$



$\boxed{\dim W_i = \text{molteplicità algebrica} = ma(c_i)}$

(n° blocchi = n° autovalori)

$$A = \begin{pmatrix} -2 & 0 & -4 & 0 \\ -1 & -1 & -1 & -2 \\ 4 & 0 & 6 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$P_\varphi(x) = (x-2)^2(x-1)^2$

$$\ker(\varphi - 2id) = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle \quad ma(2) = 2, \quad mg(2) = 1$$

$$\ker(\varphi - id) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \quad ma(1) = 2, \quad mg(1) = 1$$

<u>Filtrazione dei nuclei:</u>

$$\langle 0 \rangle \subsetneq \ker(\varphi - 2id) \subsetneq \ker(\varphi - 2id)^2 = \ker(\varphi - 2id)^3$$

DIM: $0 \quad 1 \quad 2 = 2$

$$\langle 0 \rangle \subsetneq \ker(\varphi - id) \subsetneq \ker(\varphi - id)^2 = \ker(\varphi - id)^3$$

DIM: $0 \quad 1 \quad 2 = 2$

$$M_\varphi(x) = (x-2)^2(x-1)^2 = P_\varphi(x)$$

$\Rightarrow$ esiste un vettore ciclico

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} \end{pmatrix}$$

dim(autospazio generalizzato) = = molt. algebrica

$mg \lneq ma$

$$(\varphi - 2id)^2(\varphi - id)^2 = 0 \Rightarrow (A - 2\cdot\mathbb{1})^2(A - \mathbb{1})^2 = \mathbb{0}$$

$$\operatorname{im}(A - \mathbb{1})^2 \subseteq \operatorname{Ker}(A - 2\cdot\mathbb{1})^2$$

$$(A - \mathbb{1})^2 = \begin{pmatrix} -7 & 0 & -8 & 0 \\ -1 & 0 & -1 & 0 \\ 8 & 0 & 9 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\dim(\operatorname{im}(A - \mathbb{1})^2) = 2$$

$$\dim(\operatorname{Ker}(A - 2\mathbb{1})^2) = 2$$

$$\Rightarrow \operatorname{im}(A - \mathbb{1})^2 = \operatorname{Ker}(A - 2\mathbb{1})^2$$

$w_4 \in \operatorname{Ker}(\varphi - id)^2 \setminus \operatorname{Ker}(\varphi - id)$

autovett. generalizzato di periodo 2

$w_4 = v_2$

$w_3 = (\varphi - id)(v_2) = -2v_2 + 2v_4$

autovettore relativo ad 1

$$\boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}}$$

$w_3 \; w_4$

$w_1 \; w_2$

$\mathrm{im}(\varphi - id)^2 = \ker(\varphi - 2id)^2$

$w_2 \in \ker(\varphi - 2id)^2 \setminus \ker(\varphi - 2id)$ $\quad \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}}$

$$(A - \mathbb{1})^2 = \begin{pmatrix} -7 & 0 & -8 & 0 \\ -1 & 0 & -1 & 0 \\ 8 & 0 & 9 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{im}(\varphi - id)^2 = \langle v_1 - v_3, v_2 - v_3 - v_4 \rangle$

$v_1 - v_3 \in \ker(\varphi - 2id)$

$\hookrightarrow$ non va bene come autovett. generalizzato

$w_2 = v_2 - v_3 - v_4$

$w_1 = (\varphi - 2)(v_2 - v_3 - v_4) = 4v_1 - 4v_3$

$$\hookrightarrow \begin{pmatrix} -4 & 0 & -4 & 0 \\ -1 & -3 & -1 & -2 \\ 4 & 0 & 4 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$\boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}}$

$\ker(\varphi - 2id) = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$

$\Downarrow$

w_1

$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 1 & 1 \\ & 1 \end{matrix}} \end{pmatrix}$$

$$P = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ -4 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_\varphi(x) = \begin{vmatrix} x-1 & -1 & 0 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & -1 & x-1 & -1 \\ 0 & 0 & 0 & x-1 \end{vmatrix}$$

$$= (x-1)^4$$

$$A - \mathbb{1} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\operatorname{Ker}(\varphi - id) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\underset{0}{0} < \underset{2}{\operatorname{Ker}(\varphi - id)} < \underset{4}{\operatorname{Ker}(\varphi - id)^2}$$

$$(A - \mathbb{1})^2 = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0_4$$

$$\boxed{M_\varphi = (x-1)^2}$$

$$\begin{pmatrix} \bullet & 1 & & \\ & \bullet & & \\ & & \bullet & 1 \\ & & & \bullet \end{pmatrix}$$

$w_2, w_4 \in \operatorname{Ker}(\varphi - id)^2 \setminus \operatorname{Ker}(\varphi - id)$

$$w_2 = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad w_3 = (\varphi - id)(w_4) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$w_4 = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad w_1 = (\varphi - id)(w_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

~ 0 ~ 0 ~

3×3 $\qquad P_\varphi, M_\varphi$

$$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \qquad P_\varphi(x) = (x-a)(x-b)(x-c) = M_\varphi(x)$$

$$\begin{pmatrix} a & & \\ & a & \end{pmatrix} \qquad P_\varphi(x) = (x-a)^2(x-b)$$

$$\begin{pmatrix} \ & & \\ & & \\ & & b \end{pmatrix} \quad M_\varphi(x) = (x-a)(x-b)$$

$$\begin{pmatrix} 0 & & \\ & 0 & \\ & & b-a \end{pmatrix}\begin{pmatrix} a-b & & \\ & a-b & \\ & & 0 \end{pmatrix} = O_3$$

$x-a$ in φ $\qquad$ $x-b$ in φ

$$\begin{pmatrix} a & 1 & \\ & a & \\ & & b \end{pmatrix} \quad P_\varphi(x) = (x-a)^2(x-b)$$

$$\underset{M_\varphi(x)}{\overset{P_\varphi(x)}{\parallel}}$$

$$\begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix} \quad \begin{aligned} P_\varphi(x) &= (x-a)^3 \\ M_\varphi(x) &= (x-a) \end{aligned}$$

$$\begin{pmatrix} a & & \\ & a & 1 \\ & & a \end{pmatrix} \quad \begin{aligned} P_\varphi(x) &= (x-a)^3 \\ M_\varphi(x) &= (x-a)^2 \end{aligned}$$

$$\begin{pmatrix} a & 1 & \\ & a & 1 \\ & & a \end{pmatrix} \quad \begin{aligned} P_\varphi(x) &= (x-a)^3 \\ M_\varphi(x) &= (x-a)^3 \end{aligned}$$

$\sim$ 0 $\sim$

$V \quad \dim V = 10$

$\dim[\mathrm{Ker}(\varphi-5)] = 2 < \underbrace{\dim[\mathrm{Ker}(\varphi-5)^2]}_{=3} <$

$< \dim[\mathrm{Ker}(\varphi-5)^3] = 4$

$\dim(\underbrace{\mathrm{Ker}(\varphi+2)}_{=2}) < \dim[\mathrm{Ker}(\varphi+2)^4] = 4$

$\dim(\mathrm{im}\,\varphi) = 8$

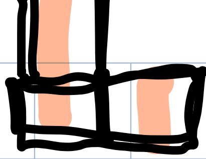
$\dim(\mathrm{Ker}\,\varphi) = \dim(V) - \dim(\mathrm{im}\,\varphi) = 2$

A.VALORI	m_a	m_g
5	4	2
-2	4	2
0	2	2

$$\boxed{P_\varphi(x) = x^2(x+2)^4(x-5)^4}$$

A.VAL $0 \rightarrow 2$ blocchetti di ordine 1

A.VAL 5 $\Leftarrow \mathrm{Ker}(\varphi-5)^3 - \mathrm{Ker}(\varphi-5)^2$



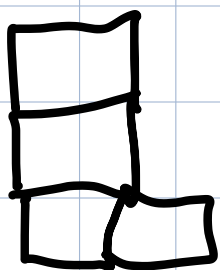
$\leftarrow \operatorname{Ker}(\varphi-5)^2 \setminus \operatorname{Ker}(\varphi-5)$

$\leftarrow \operatorname{Ker}(\varphi-5)$

$$v \in \operatorname{Ker}(\varphi-5)^3 - \operatorname{Ker}(\varphi-5)^2$$

$$(\varphi-5)(v) \in \operatorname{Ker}(\varphi-5)^2 \setminus \operatorname{Ker}(\varphi-5)$$

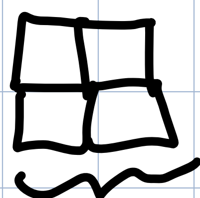
A. VAL -2



$\operatorname{Ker}(\varphi+2)^3$

$\operatorname{Ker}(\varphi+2)^2$

$\operatorname{Ker}\ \varphi+2$



$\downarrow$

un blocchetto da 3, uno da 1

due blocchetti da 2

$$M_\varphi(x) = x(x-5)^3(x+2)^3$$

$$M_\varphi(x) = x(x-5)^3(x+2)^2$$

$$\begin{pmatrix} \boxed{0} & & & & & & & & \\ & \boxed{0} & & & & & & & \\ & & \boxed{5} & & & & & & \\ & & & 5 & 1 & & & & \\ & & & & 5 & 1 & & & \\ & & & & & 5 & & & \\ & & & & & & \boxed{2} & & \\ & & & & & & & 2 & 1 \\ & & & & & & & & 2 \quad 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{0} & & & & & & & & & \\ & \boxed{0} & & & & & & & & \\ & & \boxed{5} & & & & & & & \\ & & & 5 & 1 & & & & & \\ & & & & 5 & 1 & & & & \\ & & & & & 5 & & & & \\ & & & & & & 2 & 1 & & \\ & & & & & & & 2 & & \\ & & & & & & & & 2 & 1 \\ & & & & & & & & & 2 \end{pmatrix}$$

ES

$$A = \begin{pmatrix} 1 & -6 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & -2 & -1 \end{pmatrix}$$

Soluz

$$J = \begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ & -2 \end{matrix}} & & \\ & \boxed{3} & \\ & & \boxed{\begin{matrix} 3 & 1 \\ & 3 \end{matrix}} \end{pmatrix}$$

$$P = \begin{pmatrix} 6 & 2 & -1 & -9 & 2 \\ 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 & 5 \\ 0 & -5 & 0 & 0 & 5 \\ 0 & -10 & 1 & 0 & 0 \end{pmatrix}$$

ES $\dim V = 11$

$\dim(\ker(\varphi+3)) = 2 < \dim(\ker(\varphi+3)^4) = 4$

$\dim(\ker(\varphi-1)^3) = 5$

$\dim(\operatorname{im}\varphi) = 9$

(10 possibili matrici)
(l'autovalore 1 è il principale responsabile)

ES

$$A = \begin{pmatrix} -1 & 0 & -1 & 0 & 5 \\ 0 & -5 & 0 & 0 & 3 \\ 1 & 0 & -3 & 3 & 5 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 & 1 \end{pmatrix}$$

<u>soluz</u>

$$J = \begin{pmatrix} -2 & 1 & & & \\ & -2 & & & \\ & & -2 & 1 & \\ & & & -2 & 1 \\ & & & & -2 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & -3 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ -15 & 0 & -3 & 3 & 0 \\ 0 & -5 & 0 & 0 & 1 \\ -3 & 0 & 0 & 0 & 0 \end{pmatrix}$$