Theory

1. Prove that if a series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

Set $S = \sum_{n=1}^{\infty} a_n$. By hypothesis $S \in \mathbb{R}$. Set $S_k = \sum_{n=1}^{k} a_n$.

$$\lim_{n\to+\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0$$

2. Give an example of a function which has positive derivative on its domain but is not increasing.

$$f(x) = -\frac{1}{x} \qquad f' = \frac{1}{x^2} > 0$$

<u>but</u> $f(-1) = 1 > f(1) = -1$

3. State and prove Fermat theorem on local maximums and minimums.

Theorem $f : I \to \mathbb{R}$.
x_0 is internal to I
f is differentiable on I
x_0 is a relative maximum or minimum point. Then

$$f'(x_0) = 0$$

Exercises

Exercise 1 (punti 9) Let us consider the function

$$f(x) = \sqrt{3x^2 - 2x} - \sqrt{3}x \quad ;$$

(a) determine the maximal domain D of f and the sign of $f(x)$ for every $x \in D$;

(b) compute significative limits and investigate the possibility of asymptotes;

(c) compute the derivative at the points where f is differentiable ; discuss the monotonicity of f; provided they exist, determine the infimum, the supremum, minimum and maximum points(relative and absolute);

(d) plot a qualitative graph of f.

Exercise 2 (punti 7) Let us consider the equation on $\mathbb{C}$

$$z^4 + \alpha i z^3 + 2\alpha z^2 = -12iz - 8 \qquad (\alpha \in \mathbb{R})$$

Setting $z = i$ one gets alpha=-3

(a) Determine $\alpha \in \mathbb{R}$ such that $z_0 := i$ is a solution;

(b) Verify that for the value of α found in (a), also $z_1 := 2$ is a solution;

(c) Find the remaining solutions.

Exercise 3 (punti 8) (a) Compute the limit

$$\lim_{x\to 0^+} [\cos x - 2x - 4x^2]^{\frac{1}{x}}.$$

(b) Study the character of the series (is it convergent, divergent, indeterminate?)

$$\sum_{n=1}^{+\infty} \left[\cos\left(\frac{1}{n}\right) - \frac{2}{n} - \frac{4}{n^2}\right]^{n^2}.$$

Exercise 4 (punti 8) For every $\alpha \in \mathbb{R}$ consider the function

$$f_\alpha(x) = x\,(\log(x+1))^\alpha \qquad x \in]0,1]$$

(a)For which values of the parameter $\alpha \in \mathbb{R}$ is the generalized integral

$$\int_0^1 f_\alpha(x)\,dx$$

$f_\alpha(x) = x(x + o(x))^\alpha = x^{1+\alpha}\left(1 + \frac{o(x)}{x}\right) \sim x^{1+\alpha} = \frac{1}{x^{-1-\alpha}}$

convergent if and only if $-1-\alpha < 1$, $\alpha > -2$

Ex. 3

$= \lim_{x\to 0^+} \left[1 - \frac{x^2}{2} - 2x - 4x^2\right]^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{1}{x}\log(1 - 2x + o(x))} = \lim_{x\to 0^+} e^{\frac{-2x + o(x)}{x}} = e^{-2} = \frac{1}{e^2}$

The series converges by the root test.

$\lim \sqrt[n]{a_n} = \lim\left[\cos\frac{1}{n} - \frac{2}{n} - \frac{4}{n^2}\right]^n = \frac{1}{e^2} < 1$

Exercise 2

By (a) we get $\alpha = -3$
so the equation becomes

$$P(z) = z^4 - 3iz^3 - 6z^2 + 12iz + 8 = 0$$

Since $P(2) = 0$, (b) is verified
Therefore $P(z)$ is divisible
by $(z-i)(z-2) = z^2 - (i+2)z + 2i$

Performing the division
we get

$$\frac{P(z)}{(z-i)(z-2)} = z^2 + (-2i+2)z - 4i$$

Therefore

$$P(z) = (z-i)(z-2)\left(z^2+(-2i+2)z-4i\right)$$

So we have to find the solutions of

$$z^2+(-2i+2)z-4i = 0$$

Applying the standard formula to solve 2nd degree equations we get

$$z_{2,3} = i-1+\left(\text{roots of } \Delta = (i-1)^2+4i = -1+1-2i+4i\right) =$$

$$= i-1+(\text{roots of } 2i) =$$

$$= \begin{cases} i-1+\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{2}\right) \\ i-1-\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{2}\right) \end{cases}$$

$$\begin{cases} = i-1+1+i = 2i \\ = i-1-1-i = -2 \end{cases}$$

Ex 4

$f(x) = \sqrt{3x^2-2x} - \sqrt{3}\,x$

Domain $= \{x\in\mathbb{R} : 3x^2-2x\} =$

$=]-\infty, 0] \cup [\frac{2}{3}, +\infty[= D$

Sign . $f(x) \geq 0 \iff \sqrt{3x^2-2x} \geq \sqrt{3}\,x$

$\iff \left(\underbrace{\begin{cases} x \geq 0 \\ \cancel{3x^2} - 2x \geq \cancel{3x^2} \end{cases}}_{\{0\}} \cup \{x \leq 0\} \right) \cap D$

$= \{x \leq 0\}$ —

The function is continuous at every $x \in D$

Limits

$\lim_{x\to+\infty} \sqrt{3x^2-2x} - \sqrt{3}\,x = +\infty - \infty =$

$\lim_{x\to+\infty} \frac{\cancel{3x^2} - 2x - \cancel{3x^2}}{\sqrt{3x^2-2x}+\sqrt{3}\,x} = \lim_{x\to+\infty} \frac{-2x}{x\left(\sqrt{3-\frac{2}{x}}+\sqrt{3}\right)} =$

$= -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$

$\lim_{x\to-\infty} f(x) = +\infty + \infty = +\infty$

Asymptote at $-\infty$?

$m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{3x^2-2x} - \sqrt{3}\,x}{x} =$

$$\lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{3-\frac{2}{x}} - \sqrt{3}\,\cancel{x}}{\cancel{x}} = -2\sqrt{3}$$

$$p = \lim_{x\to-\infty} f(x) + 2\sqrt{3}\,x = \sqrt{3x^2-2x} - \sqrt[3]{x} + 2\sqrt{3}\,x$$

$$= \lim_{x\to-\infty} \underbrace{\sqrt{3x^2-2x}}_{\to+\infty} + \underbrace{\sqrt{3}\,x}_{\to-\infty} = +\infty - \infty =$$

$$= \frac{-2\cancel{x}}{\cancel{x}\left(-\sqrt{3-\frac{2}{x}} - \sqrt{3}\right)} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$y = -2\sqrt{3}\,x + \frac{1}{\sqrt{3}}$ is a oblique asymptote.

Derivative:

$\forall x \in]-\infty, 0[\cup]\frac{2}{3}, +\infty[= \mathring{D}$

$$f'(x) = \frac{6x-2}{2\sqrt{3x^2-2x}} - \sqrt{3} = \boxed{\frac{3x-1}{\sqrt{3x^2-2x}} - \sqrt{3}}$$

Monotonicity

$$f'(x) \geq 0 \iff \left\{3x-1 \geq \sqrt{3x^2-2x}\cdot\sqrt{3}\right\} \cap \mathring{D}$$

$$\iff \left\{\begin{array}{l} 3x-1 \geq 0 \\ \cancel{9x^2} - \cancel{6x} + 1 \geq \cancel{9x^2} - \cancel{6x} \end{array}\right\} \cap \mathring{D}$$

$$\iff \left\{x \geq \frac{1}{3}\right\} \cap \mathring{D} = \left\{x > \frac{2}{3}\right\}$$

$\Rightarrow$ The function is (strictly) increasing on $]\frac{2}{3}, +\infty[$ $\underset{\text{by continuity}}{\Longrightarrow}$ on $[\frac{2}{3}, +\infty[$

and strictly decreasing on $]-\infty, 0[$ $\underset{\text{by continuity}}{\Longrightarrow}$ on $]-\infty, 0]$.

Right derivative at $x = \frac{2}{3}$?

No:

$$\lim_{x \to \frac{2}{3}^+} = \frac{3x-1}{\sqrt{3x^2-2x}} - \sqrt{3} = +\infty$$

(the tangent is vertical)

Left derivative at 0 ?

No:

$$\lim_{x \to 0} f'(x) = -\infty$$

(the tangent is vertical)