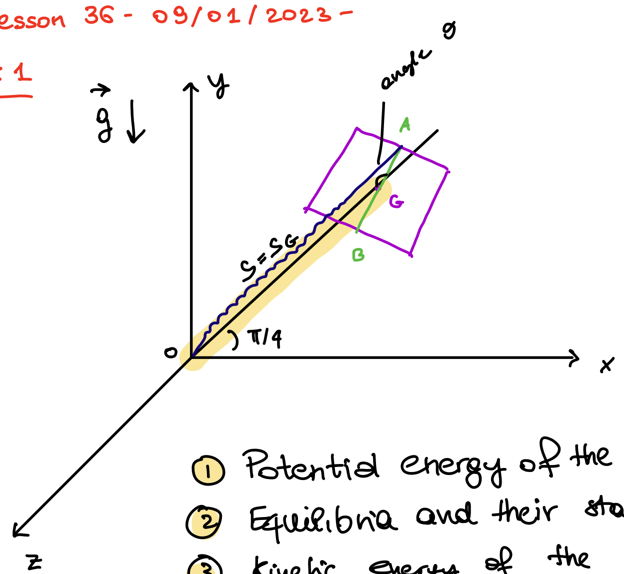
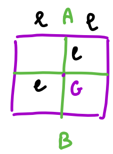
Lesson 36 - 09/01/2023 -

Ex 1



2ℓ, m on the plane Oxy



middle point

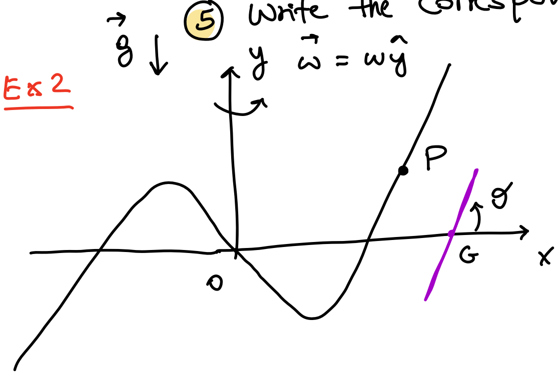
Spring between O and A.

Spring constant $h > 0$.

suppose $\lambda \neq 1$

1. Potential energy of the system.
2. Equilibria and their stab. depending on $\lambda = mg\sqrt{2}/2h\ell$.
3. Kinetic energy of the system.
4. Frequencies of small oscillations around a stable eq.
5. Write the corresponding Hamiltonian.

Ex 2



$y = f(x) = x^3 - x$

$OP(x) = (x, f(x))$

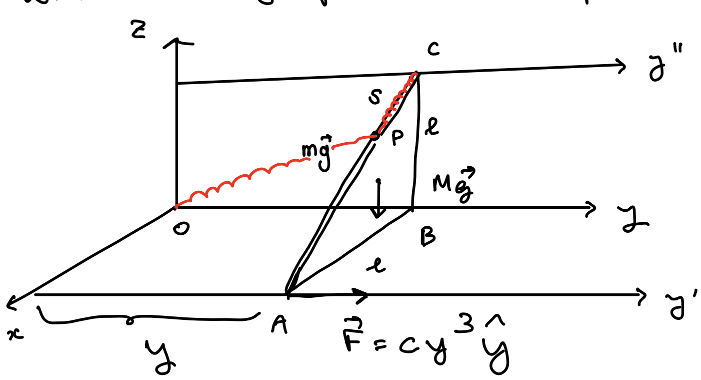
Point P of mass m on the cubic constraint.

1. Det, V, K and L. Lagrangian component of the Coriolis force?
2. Suppose $\omega = 0$ but there is a bar with the baricenter on the x-axis. m, ℓ.

Use Lagrangian coordinates $x = x_P$
$s = x_G$
θ as in figure

write new Lagrangian and interpret conserved quantities.

Ex 3

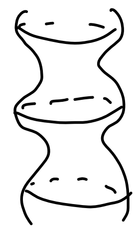


$AB = BC = \ell$

1. Equilibria and their stability.
2. Frequencies of small oscillations around a stable equilibrium.

Ex 4 Point P of mass 1 moving on a surface of rotation with parametric eqs:

$$\begin{cases} x = (3+\cos\theta)\cos\varphi \\ y = (3+\cos\theta)\sin\varphi \\ z = \theta \end{cases}$$



where $\theta \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$

1. L and L_R
2. Phase-portrait of the reduced system. Equilibria and stability of the reduced system. Description (not in detail) of the dynamics of the original system.

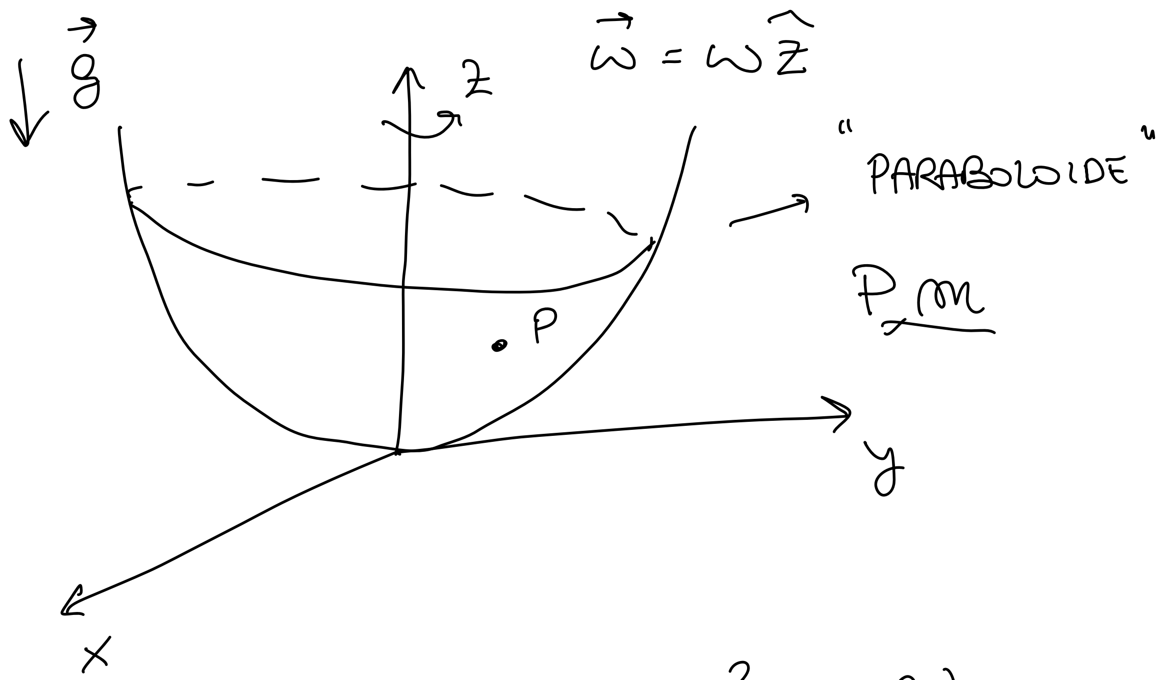
Ex 5 Consider the v.f.

$$X = \begin{pmatrix} -x-y+xy \\ 5x+y-\frac{1}{2}y^2 \end{pmatrix}$$

1. Linearize around $(0,0)$. Stability of $(0,0)$ → $\lambda = \pm i\sqrt{2}$ with spectral method (for the reduced system).
2. Infos on the stability of $(0,0)$ for the original system? NO
3. Let consider $W_a = 5x^2+y^2+2xy+axy^2$, $a \in \mathbb{R}$. Det. $a \in \mathbb{R}$ s.t. W_a is a first integral. $\underline{a=-1}$
4. Can W_a be used in order to study the stability of the origin?

$$\text{Hess } W_{-1}(0,0) = \begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$$

Ex 6

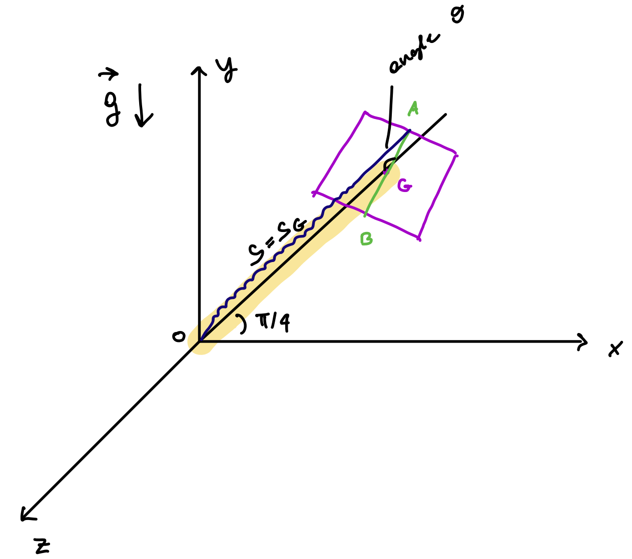


$$z = z(x,y) = \frac{1}{2}(ax^2 + by^2)$$

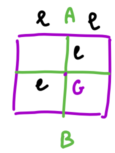
Suppose $b > a \, (> 0)$

1. Write V (for the conservative forces)
   Write Q_x, Q_y for the Coriolis force
   Write K and the corresponding $Q(x,y)$.
2. For which conditions on m, g, a, b, ω, $(0,0)$ is the unique equilibrium? And when stable?
3. If $\exists$ a viscous force $\vec{F} = -K\dot{\vec{OP}}$, $K > 0$, the conditions given above are still valid?

Ex 1 - Solution -



2ℓ, m on the plane Oxy



middle point

Spring between O and A.

Spring constant $h > 0$.

$$V = V_{gr} + V_{el} = mg\frac{\sqrt{2}}{2}s + \frac{1}{2}h\left(s^2 + \ell^2 - 2\ell s\cos\theta\right)$$

$$\begin{cases} V_s = mg\sqrt{2}/2 + hs - h\ell\cos\theta = 0 \\ V_\theta = h\ell s\sin\theta = 0 \end{cases}$$

$$V_\theta = 0 \quad \to \quad s = 0, \quad \theta = 0, \quad \theta = \pi$$

$$V_s = 0 \implies hs = h\ell\cos\theta - mg\sqrt{2}/2$$

$$\implies s = \ell\cos\theta - \frac{mg\sqrt{2}}{2h} = \ell\left(\cos\theta - \underbrace{\boxed{\frac{mg\sqrt{2}}{2h\ell}}}_{=\lambda}\right) =$$

$$= \ell(\cos\theta - \lambda) \qquad (s, \theta)$$

$$P_1 = (\ell(1-\lambda), 0)$$

$$P_2 = (-\ell(1+\lambda), \pi)$$

$$\left.\begin{array}{l} P_3 = (0, \arccos\lambda) \\ P_4 = (0, -\arccos\lambda) \end{array}\right] \quad \text{IF } \lambda < 1$$

$$H_V(s,\theta) = \begin{pmatrix} h & h\ell\sin\theta \\ h\ell\sin\theta & h\ell s\cos\theta \end{pmatrix} \qquad H_V(P_1) = \begin{pmatrix} h & 0 \\ 0 & h\ell^2(1-\lambda) \end{pmatrix}$$

$$\det H_V(s,\theta) = h^2 \ell s\cos\theta - h^2 \ell^2 \sin^2\theta$$

$$\det(P_1) = h^2\ell^2(1-\lambda) > 0$$ if $1-\lambda > 0 \iff \lambda < 1$ STABLE

if $\lambda > 1$ UNSTABLE

$$\det(P_2) = h^2\ell\,(-\ell(1+\lambda))(-1) = +h^2\ell^2(1+\lambda)$$ STABLE

$$\det(P_3) = -h^2\ell^2 \underbrace{\sin^2(\arccos\lambda)}_{>0} < 0$$ UNSTABLE

$$\det(P_4) = -h^2\ell^2 \sin^2(-\arccos\lambda) < 0$$ UNSTABLE.

Kinetic energy (Konig Theorem).

$$K = \frac{1}{2} m |\vec{v}_G|^2 + \frac{1}{2}\vec{\omega}\cdot I_G\vec{\omega}$$

$$= \frac{1}{2} m \dot{s}^2 + \frac{1}{2}\underbrace{\left[\frac{2}{3} m\ell^2\right]}\dot{\theta}^2$$

$$= \frac{1}{12} m(4\ell^2 + 4\ell^2) = \frac{\cancel{8}^{\,2}}{\cancel{12}_{\,3}} m\ell^2$$

$$A(s,\theta) = \begin{pmatrix} m & 0 \\ 0 & \frac{2}{3}m\ell^2 \end{pmatrix}$$ diagonal!

Freq. of small oscillations around P_1 (when stable) $\to \lambda < 1$

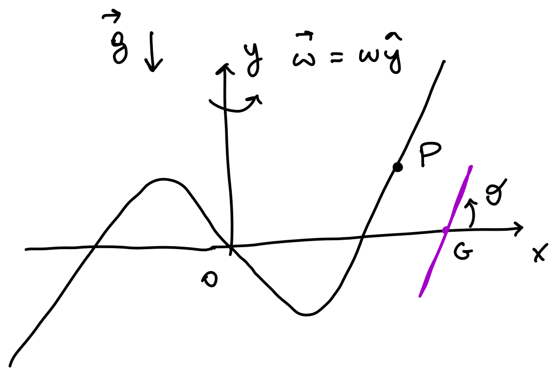
$$\omega_1^2 = \frac{h}{m}, \quad \omega_2^2 = \frac{h\cancel{\ell^2}(1-\lambda)\cdot 3}{2m\cancel{\ell^2}} = \frac{3h(1-\lambda)}{2m}$$

Hamiltonian?

$$L = K - V \Rightarrow H = \frac{1}{2} A^{-1} p\cdot p + V$$

easy to compute since $A(s,\theta)$ is diagonal!

Ex 2 - Solution -



$y = f(x) = x^3 - x$

$\vec{OP}(x) = (x, f(x))$

Point P of mass m on the cubic constraint.

$$V = V_{gr} + V_{cf} = mg\, f(x) - \frac{\omega^2}{2} m x^2 =$$

$$= mg\,(x^3 - x) - \frac{\omega^2}{2} m x^2 = V(x)$$

Kinetic energy ?

$\vec{OP}(x) = (x, f(x))$

$\vec{v}_P(x, \dot{x}) = (\dot{x}, f'(x)\dot{x})$

$$K_P(x, \dot{x}) = \frac{1}{2} m |\vec{v}_P|^2 =$$

$$= \frac{1}{2} m (\dot{x}^2 + (f'(x)\dot{x})^2) =$$

$$= \frac{1}{2} m (1 + \underbrace{(f'(x))^2}_{(3x^2-1)^2}) \dot{x}^2$$

$L = K - V$

Recall that the Lagr. component of the Coriolis force $f = -2m\,\vec{\omega} \wedge \vec{v}_P$ is $\equiv 0$ since the vectors $\vec{\omega}$, $\vec{v}_P$ and "virtual displacement" are in the same plane Oxy.

$\underline{\omega \equiv 0}$ + BAR.

Note that we need to use an additional lagrangian parameter $s = x_G$. ($x_G \neq x_P$!!)

When $\omega \equiv 0$ then $V = V_{gr}$ (of the point) + const.

$$K_{NEW} = K_P + K_{BAR} =$$

König

$$= \frac{1}{2} m \left[1 + (f'(x))^2 \right] \dot{x}^2 + \frac{m}{2} \dot{s}^2 + \frac{1}{2} \frac{m\ell^2}{12} \dot{\theta}^2$$

$$L = K_{NEW} - V = K_{NEW} - mg f(x)$$

$$L(x, \underbrace{s, \theta}, \dot{x}, \dot{s}, \dot{\theta})$$

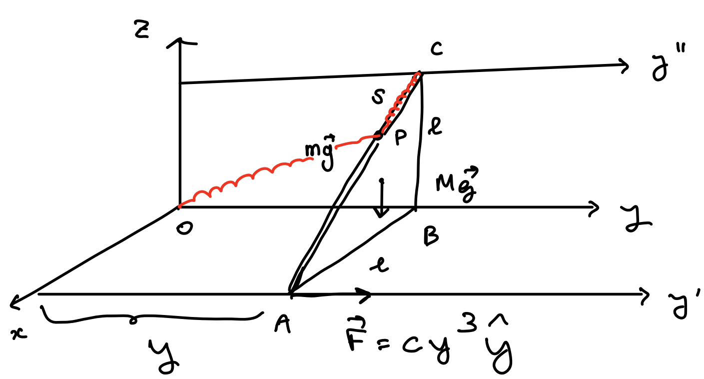
cyclic coo.

$$\frac{\partial L}{\partial \dot{s}} = m \dot{s} \longrightarrow \text{QUANTITY of MOTION of the BAR.}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{m\ell^2}{12} \dot{\theta} \longrightarrow$$ ANGULAR MOMENTUM of the BAR wrt the BARICENTER.

$$E = K + V$$

Ex 3 - Solution -



AB = BC = ℓ

$$\vec{OP}(y, s) =$$

$$= \frac{s}{\sqrt{2}}\hat{x} + y\hat{y} + \left(\ell - \frac{s}{\sqrt{2}}\right)\hat{z}$$

$$\Downarrow$$

$$\vec{v}_P = \frac{\dot{s}}{\sqrt{2}}\hat{x} + \dot{y}\hat{y} \left(-\frac{\dot{s}}{\sqrt{2}}\right)\hat{z}$$

$$|\vec{v}_P| = \underline{\dot{s}^2 + \dot{y}^2}$$

$$dL^F = \vec{F} \cdot dA =$$

$$= c y^3 \hat{y} \cdot \underline{dy\,\hat{y}} =$$

$$= c y^3 dy = -d\left(-\frac{c y^4}{4}\right)$$

To write the elastic potential of the spring connecting O and P, we compute:

$$|OP(y, s)|^2 =$$

$$= \frac{s^2}{2} + y^2 + \ell^2 + \frac{s^2}{2} - \frac{2\ell s}{\sqrt{2}}$$

$$= s^2 + y^2 - \frac{2\ell s}{\sqrt{2}} + \text{const.}$$

$$\Downarrow$$

$$U(y,s) =$$

$$= \frac{1}{2} h \left[s^2 + y^2 - \frac{2\ell s}{\sqrt{2}} \right] + \frac{1}{2} h s^2 -$$

$$- \frac{c y^4}{4} + mg\left(\ell - \frac{s}{\sqrt{2}} \right) =$$

$$= h s^2 + \frac{1}{2} h y^2 - \frac{c y^4}{4} - \frac{h \ell s}{\sqrt{2}} -$$

$$- \frac{mgs}{\sqrt{2}} =$$

$$= \frac{h y^2}{2} - \frac{c y^4}{4} + h s^2 - \frac{1}{\sqrt{2}} s \left(h\ell + mg \right)$$

—x—x—

We write now the Kinetic energy.

$$|\vec{v}_p|^2 = \dot{s}^2 + \dot{y}^2.$$

$$K = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} m \left(\dot{s}^2 + \dot{y}^2 \right)$$

$$= \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{2} m \dot{s}^2$$

Equilibria

$$\begin{cases} U_y = hy - cy^3 = 0 \\ U_s = \underline{2hs - \frac{\sqrt{2}}{2}(he + mg)} = 0 \end{cases}$$

$y = 0$ OR $h - cy^2 = 0$

$y^2 = \frac{h}{c}$, $y = \pm\sqrt{\frac{h}{c}}$ $(c > 0)$

$$\cancel{2h} s = \frac{\sqrt{2}}{2 \cdot 2h}(he + mg)$$

$$\bar{s} = \frac{\sqrt{2}}{4h}(he + mg)$$

$E_1 = (0, \bar{s})$

$E_2 = \left(\sqrt{\frac{h}{c}}, \bar{s}\right)$

$E_3 = \left(-\sqrt{\frac{h}{c}}, \bar{s}\right)$

$$\mathrm{Hess}\,\mathcal{U} = \begin{pmatrix} h - 3cy^2 & 0 \\ 0 & 2h \end{pmatrix}$$

$$\mathrm{Hess}(E_1) = \begin{pmatrix} h & 0 \\ 0 & 2h \end{pmatrix} \qquad \text{STABLE}$$

$$\mathrm{Hess}(E_2) = \begin{pmatrix} h - 3\cancel{c}\frac{h}{\cancel{c}} & 0 \\ 0 & 2h \end{pmatrix} \qquad \text{UNSTABLE}$$

$$\mathrm{Hess}(E_2) = \mathrm{Hess}(E_3)$$

For the Non def. Hess. Theo!

Small oscillations around E_1

$$\det\left(\mathrm{Hess}\,\mathcal{U}|_{E_1} - \omega^2 Q|_{E_1}\right) =$$

$$= \det\left[\begin{pmatrix} h & 0 \\ 0 & 2h \end{pmatrix} - \omega^2 \begin{pmatrix} M+m & 0 \\ 0 & m \end{pmatrix}\right] = 0$$

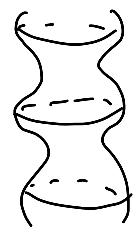
$$\omega_1 = \sqrt{\frac{h}{M+m}}\,, \quad \omega_2 = \sqrt{\frac{2h}{m}}$$

—x—x—

Ex 4 – Solution –

Point P of mass 1 moving on a surface of rotation with parametric eqs:

$$\begin{cases} x = (3+\cos\vartheta)\cos\varphi \\ y = (3+\cos\vartheta)\sin\varphi \\ z = \vartheta \end{cases}$$



where $\vartheta \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$

Since we have a "spontaneous" motion,

$L = K$.

$$\begin{cases} \dot{x} = -\dot{\theta}\sin\vartheta\cos\varphi - (3+\cos\theta)\dot{\varphi}\sin\varphi \\ \dot{y} = -\dot{\theta}\sin\vartheta\sin\varphi + (3+\cos\theta)\dot{\varphi}\cos\varphi \\ \dot{z} = \dot{\vartheta} \end{cases}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\theta}^2(1+\sin^2\theta) + (3+\cos\theta)^2\dot{\varphi}^2$$

$$L = K = \frac{1}{2}(1+\sin^2\theta)\dot{\theta}^2 + \frac{1}{2}(3+\cos\theta)^2\dot{\varphi}^2$$

φ is a cyclic coo.

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (3+\cos\theta)^2\dot{\varphi}$$

$$\Rightarrow \dot{\varphi} = \frac{p_\varphi}{(3+\cos\theta)^2}$$

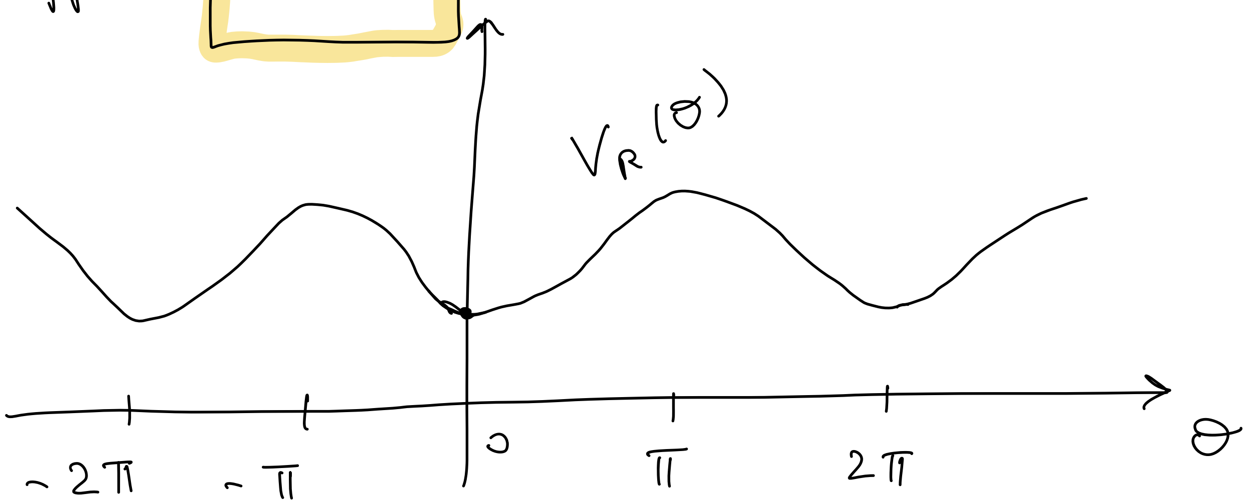
$$\Rightarrow \dot{\varphi} = c \big/ (3+\cos\theta)^2$$

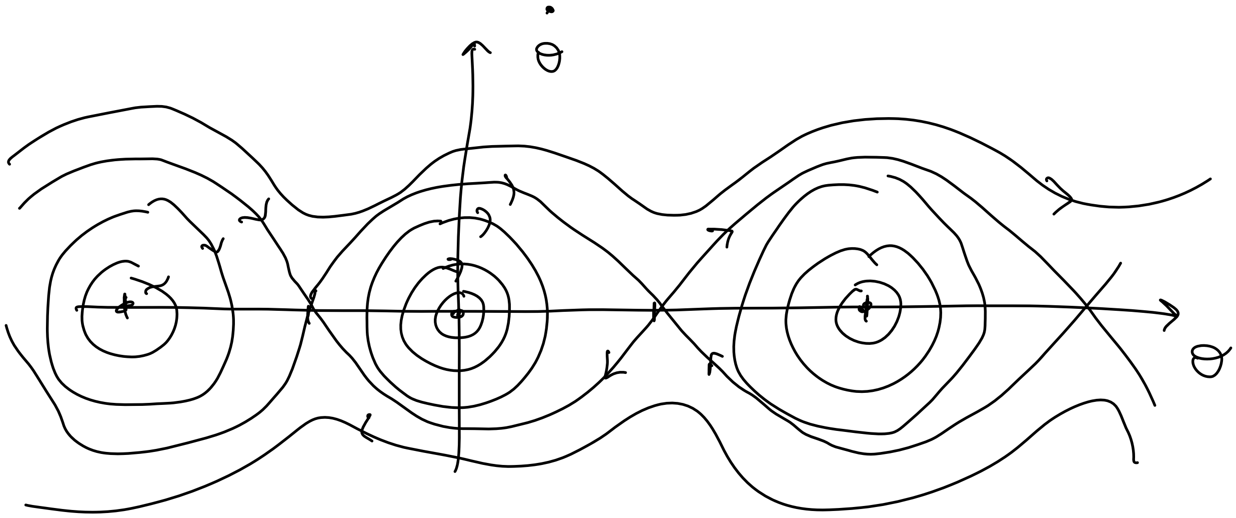
$$L_R(\theta,\dot{\theta}) =$$

$$= \frac{1}{2}(1+\sin^2\theta)\dot{\theta}^2 - \frac{1}{2}(3+\cos\theta)^2\dot{\varphi}^2 \Big|_{\dot{\varphi}}$$

$$= \frac{1}{2}(1+\sin^2\theta)\dot{\theta}^2 - \underbrace{\frac{c^2}{2(3+\cos\theta)^2}}_{V_R(\theta)}$$

Suppose $\boxed{c \neq 0.}$





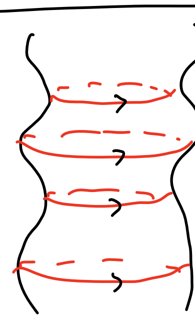
Phase-portrait of the reduced system.

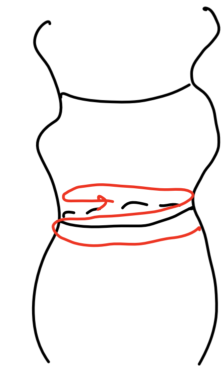
Eq. $\vartheta = k\pi$, $k \in \mathbb{Z}$

stable if $k = 0, \pm 2, \pm 4 \dots$

$k = \pm 1, \pm 3 \dots$ unstable.

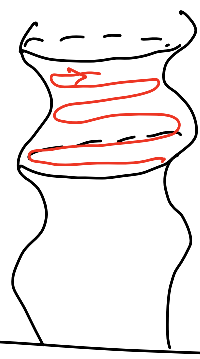
Original motions.

REDUCED SYSTEM	ORIGINAL SYSTEM
Equilibria	 periodic orbits on equators. max $\to$ UNST. min $\to$ STAB.
oscillations around a STABLE eq.	?



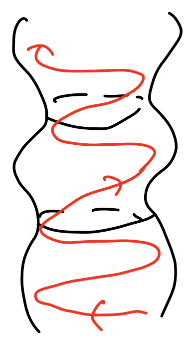
BOUNDED MOTION CLOSE TO A MINIMAL EQUATOR.

Separatrix



ASYMPTOTIC ORBIT BETWEEN 2 MAXIMAL EQUATORS

ORBITS NOT BOUNDED



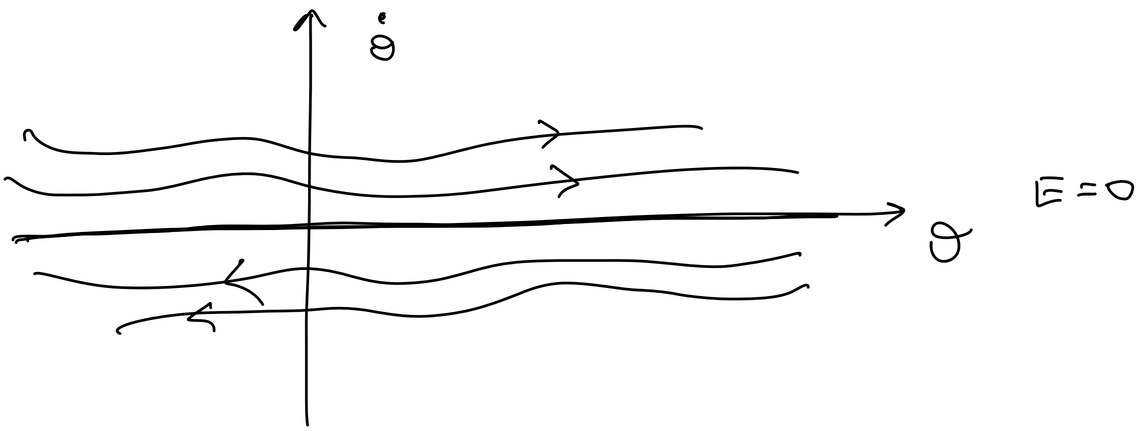
Spirals on the whole surface

$\boxed{C=0}$

$$\frac{1}{2}(1+\sin^2\theta)\dot{\theta}^2 = E \Rightarrow \dot{\theta}^2 = \frac{2E}{1+\sin^2\theta}$$

$(E \geq 0)$

$$\Rightarrow \dot{\theta} = \pm\sqrt{\frac{2E}{1+\sin^2\theta}}$$

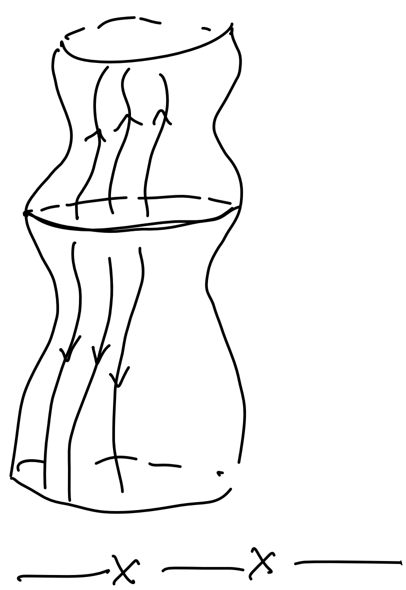


Equilibria $\theta \in \mathbb{R}$, all unstable (the unstability of the free particle, essentially).

On the complete system:

$$\dot\varphi = \frac{c=0}{(3+\cos\theta)^2} = 0$$

$\dot\varphi = 0$



Ex 6 - solution -

$$K = K(x, y, \dot{x}, \dot{y}) =$$

$$= \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + (ax\dot{x} + by\dot{y})^2 \right] =$$

$$= \frac{1}{2} (\dot{x}, \dot{y}) \underbrace{\begin{pmatrix} m(1+a^2x^2) & mabxy \\ mabxy & m(1+b^2y^2) \end{pmatrix}}_{m \begin{pmatrix} (1+a^2x^2) & abxy \\ abxy & (1+b^2y^2) \end{pmatrix}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$V = \frac{mg}{2} (ax^2 + by^2) +$$

$$+ \left(-\frac{m}{2} \omega^2 (x^2 + y^2) \right)$$

$$dL^{\text{coriolis}}\Big|_{\text{prob}} =$$

$$= -2m\vec{\omega} \wedge \begin{pmatrix} \dot{x} \\ \dot{y} \\ ax\dot{x} + by\dot{y} \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ ax\,dx + by\,dy \end{pmatrix}$$

$$= \det \begin{pmatrix} dx & dy & ax\,dx + by\,dy \\ 0 & 0 & -2m\omega \\ \dot{x} & \dot{y} & ax\dot{x} + by\dot{y} \end{pmatrix}$$

$$= -(-2m\omega)(\dot{y}\,dx - \dot{x}\,dy) =$$

$$= \underbrace{2m\omega\dot{y}}_{Q_x}\,dx \underbrace{- 2m\omega\dot{x}}_{Q_y}\,dy$$

EQUILIBRIA

IT IS POSSIBLE TO USE ONLY L-D Theo.

$$\begin{cases} 0 = V_x = m(ga - \omega^2)x \\ 0 = V_y = m(gb - \omega^2)y \end{cases}$$

$(0,0)$ is the unique eq. IFF

$$\begin{cases} ga - \omega^2 \neq 0 \\ gb - \omega^2 \neq 0. \end{cases}$$

$$\text{Hess } V(0,0) = m\begin{pmatrix} ga - \omega^2 & 0 \\ 0 & gb - \omega^2 \end{pmatrix}$$

SUFF. COND. (L-D. theo !!)

$$\begin{cases} ga - \omega^2 > 0 \\ gb - \omega^2 > 0 \end{cases} \quad (*)$$

+ diss force : The conditions given by using the L-D theo. are still true. (0,0) on conditions (*) is still stable also with the dissipative force

$$\vec{F} = -k\,\vec{v}_p \;!!$$

X ———————————————— X