$$\sum_{n=1}^{\infty} \frac{\left(x - \sin\frac{1}{n}\right)^n}{\boxed{1 + n(x+4)^2}}$$

Study convergence for every $x \geq 0$

$x > 0$

$x - \sin\frac{1}{n} \geq 0$

$\Updownarrow$

$x \geq \sin\frac{1}{n}$

true for $\forall n \geq \bar{n}$ for $\bar{n}$ sufficiently large. (because $\lim \sin\frac{1}{n} = 0$)

if $x = 0$

$$\sum \frac{\left(-\sin\frac{1}{n}\right)^n}{1 + n(x+4)^2}$$

it is an alternating series

$x > 0$

$$\sum \frac{\left(x - \sin\frac{1}{n}\right)^n}{1 + n(x+4)^2}$$

Positive series for $n \geq \bar{n}$

Apply root test

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{\left(x - \sin\frac{1}{n}\right)^n}{1 + n(x+4)^2}} =$$

$$\lim_{n \to +\infty} \frac{x - \sin\frac{1}{n}}{\sqrt[n]{1 + n(x+4)^2}} = \frac{x - \sin\frac{1}{n}}{e^{\log\left(\left(1 + n(x+4)^2\right)^{\frac{1}{n}}\right)}} \longrightarrow = \boxed{x}$$

because

$$\lim_{n \to +\infty} \log\left(\left(1 + n(x+4)^2\right)^{\frac{1}{n}}\right) =$$

$$\lim_{n \to +\infty} \frac{1}{n} \log\left(1 + n(x+4)^2\right) = 0$$

The series converges if $0 < x < 1$

" " diverges if $x > 1$

$\boxed{x = 1}$

$$\sum \frac{\left(1 - \sin\frac{1}{n}\right)^n}{1 + 25n}$$

$$\frac{\left(1-\sin\frac{1}{n}\right)^n}{1+25n} \sim \frac{1}{n} \ ?$$

$$\lim_{n\to 0} \frac{\left(1+\sin\frac{1}{n}\right)^n}{1+25n} = \frac{1}{n}$$

$$\lim_{n\to+\infty} \frac{\left(1-\sin\frac{1}{n}\right)^n}{1+25n}\, n =$$

$$= \lim_{n\to+\infty} \frac{n}{1+25n} \cdot \left(1-\sin\frac{1}{n}\right)^n$$

$$\frac{1}{25} \quad = \frac{1}{25} \lim e^{n\left(\log\left(1-\sin\frac{1}{n}\right)\right)}$$

$$= \frac{1}{25} \lim e^{n\left(-\sin\frac{1}{n} + o\left(\frac{1}{n}\right)\right)} = (*)$$

$$\lim n\left(-\sin\frac{1}{n} + o\left(\frac{1}{n}\right)\right) =$$

$$= -\boxed{\lim n \sin\frac{1}{n}} = -\lim \frac{\sin\frac{1}{n}}{\frac{1}{n}} =$$

$= -1$

$(*)\; = \frac{1}{25} \cdot \frac{1}{e} = \frac{1}{25e}$

So for $x=1$ our series is asymt. to the harmonic series ~~so~~ it doesn't converge.

$x = 0 \quad \sum \frac{\left(-\sin\frac{1}{n}\right)^n}{1+n(x+3)^2} =$

$= \sum (-1)^n \frac{\left(\sin\frac{1}{n}\right)^n}{1+n16} =$

$= \sum (-1)^n a_n \qquad a_n = \frac{\left(\sin\frac{1}{n}\right)^n}{1+16n}$

Try Liebniz:

$\lim a_n = 0$?

$a_{n+1} \leq a_n \quad \forall n$?

$\lim_{n\to+\infty} \frac{\left(\sin\frac{1}{n}\right)^n}{1+16n} = 0$ o.k.

$$a_n = \frac{\left(\sin\frac{1}{n}\right)^n}{1+16n} \geq \frac{\left(\sin\frac{1}{n+1}\right)^n}{1+16n} \geq$$

$$\geq \frac{\left(\sin\frac{1}{n+1}\right)^{n+1}}{1+16n} \geq \frac{\left(\sin\frac{1}{n+1}\right)^{n+1}}{1+16(n+1)}.$$

$$= a_{n+1}$$

o.k

(a_n is decreasing)

All Leibniz hypotheses are satisfied

$\Longrightarrow$ the series converges.

Exercise:

$$f(x) := 1 + \underbrace{\int_0^x \frac{1}{(\sin t + 4)^2}\,dt}$$

$f(0) = 1$

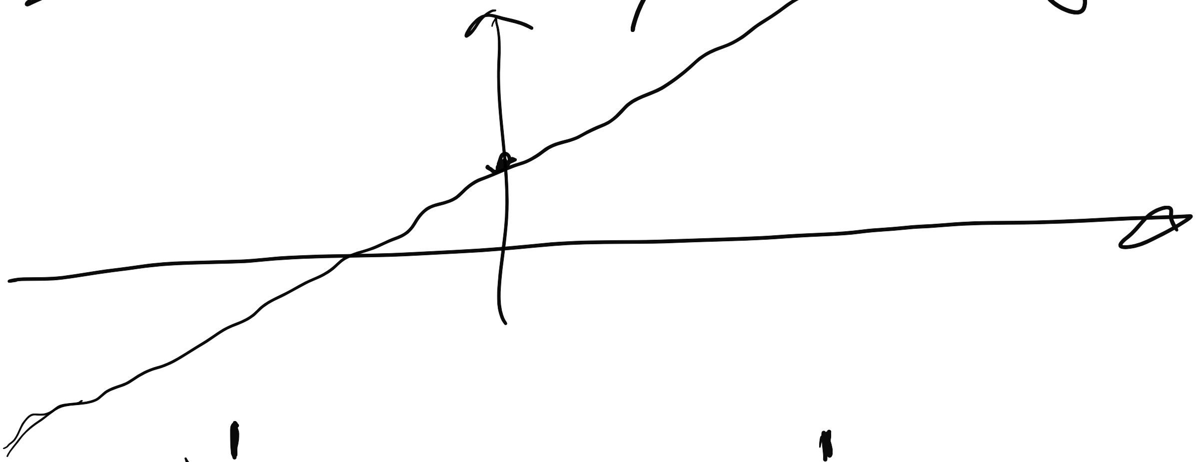
Prove that f is invertible and find $(f^{-1})'(1)$

Domain of $f = \mathbb{R}$

$$f'(x) = \frac{d}{dX}\left(\int_0^x \frac{1}{(\sin t+4)^2}\,dt\right) =$$

$$= \frac{1}{(\sin x+4)^2} > 0$$

$\Longrightarrow$ f is strictly increasing



$$(f^{-1})'(1) = (f^{-1})'(f(0)) =$$

$$= \frac{1}{f'(0)} = \frac{1}{1/16} = 16$$

$$\boxed{(f^{-1})'(f(x)) = \frac{1}{f'(x)}}$$

Continue the study of f.
(Sign, asymptots, limits)

Study

$f(x) := \arctan\left(\frac{|x+1|}{x^2+9}\right)$

Domain $= \mathbb{R}$

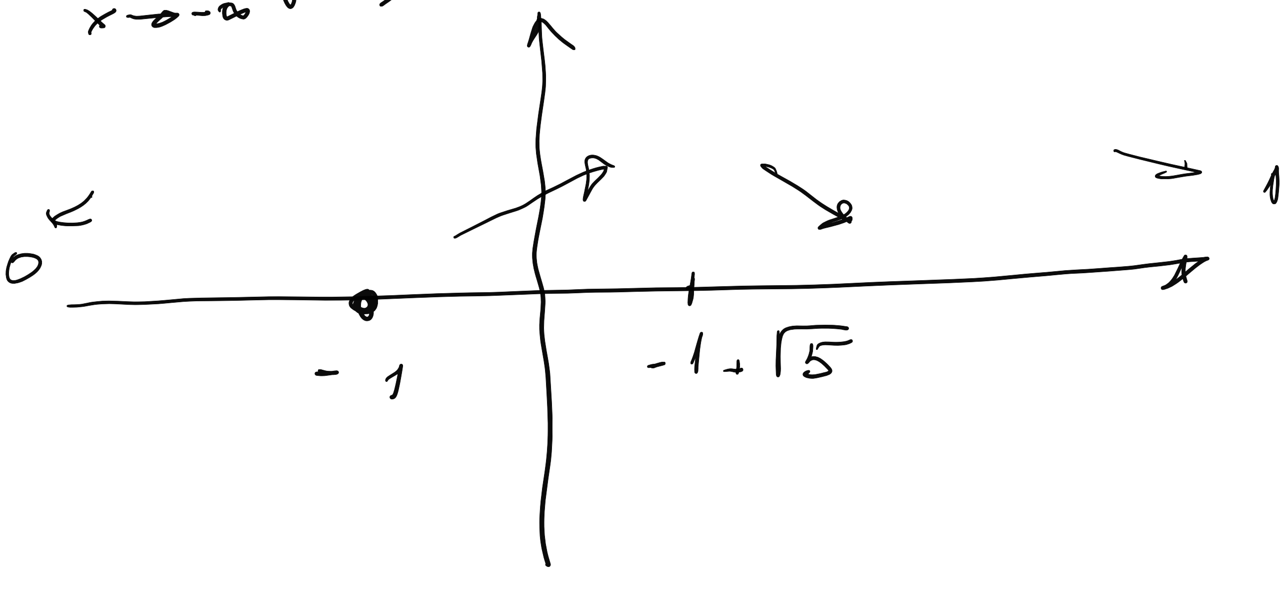
Sign of f: $f(x) \geq 0$

$\iff \frac{|x+1|}{x^2+9} \geq 0 \quad \forall x$

$f(x) = 0 \iff x = -1$

$\lim_{x \to +\infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$



$$|x+1| = \begin{cases} x+1 & x \geq -1 \\ -x-1 & x \leq -1 \end{cases}$$

for $\boxed{x > -1}$

$$f'(x) = \left(\arctan\left(\frac{x+1}{x^2+4}\right)\right)' =$$

$$\frac{1}{1+\frac{(x+1)^2}{(x^2+4)^2}} \cdot \frac{x^2+4-2x(x+1)}{(x^2+4)^2} =$$

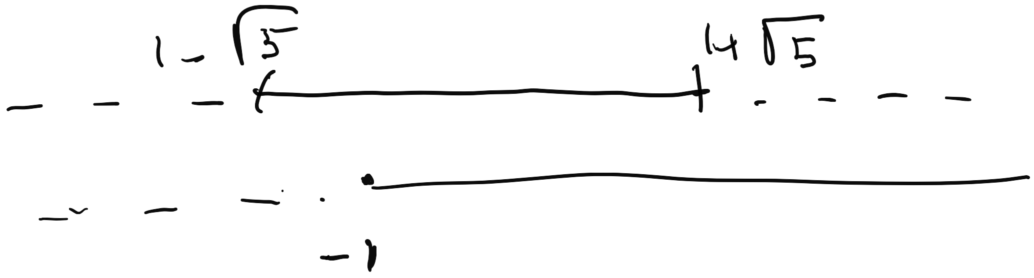
$$\frac{\cancel{(x^2+4)^2}}{(x^2+4)^2+(x+1)^2} \cdot \frac{-x^2+4-2x}{\cancel{(x^2+4)^2}} \geq 0$$

$$\Longleftrightarrow \begin{cases} -x^2+4-2x \geq 0 \\ x > -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2+2x-4 \leq 0 \\ x > -1 \end{cases}$$

$$x_{1,2} = 1 \pm \sqrt{1+4} = 1 \pm \sqrt{5}$$

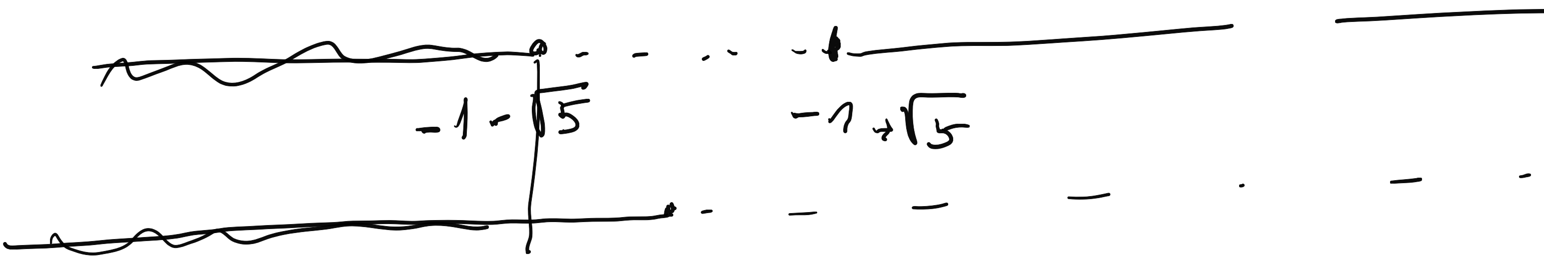
$$\begin{cases} 1-\sqrt{5} \leq x \leq 1+\sqrt{5} \\ x > -1 \end{cases}$$



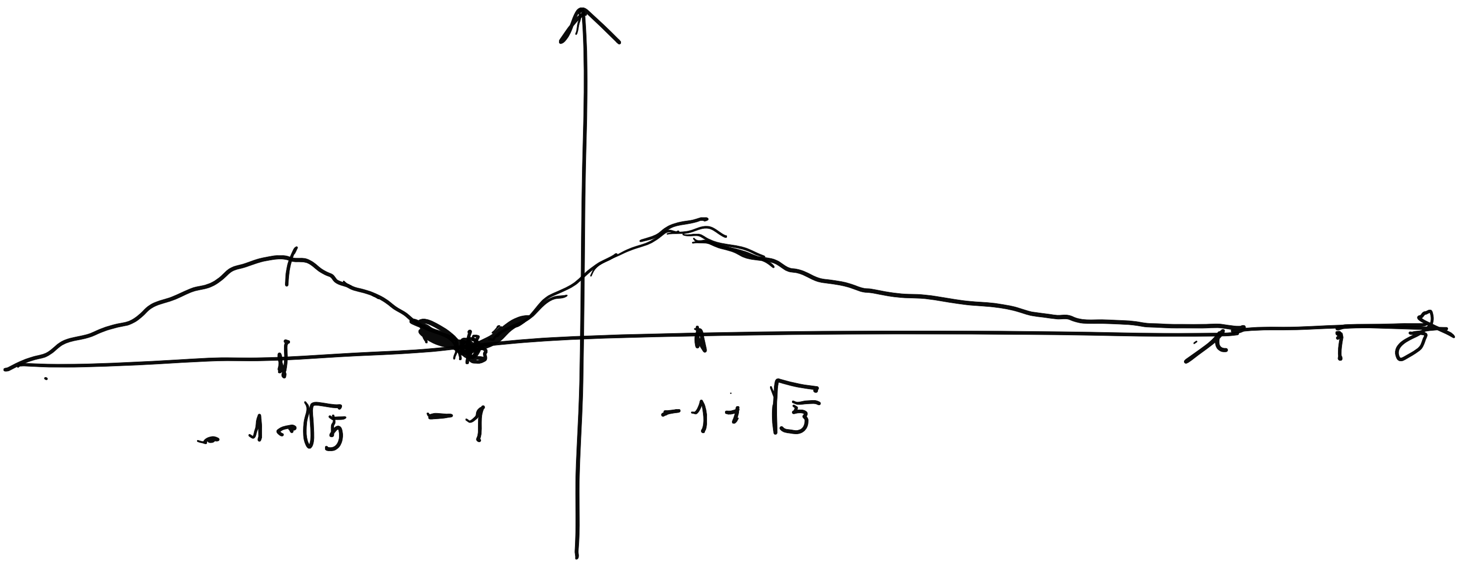
$$-1 < x \leq 1+\sqrt{5}$$

$$\begin{cases} f'(x) = \left(\dfrac{-x-1}{x^2+4}\right)' \geq 0 \\ x < -1 \end{cases}$$

$$\begin{cases} x \leq -1-\sqrt{5} \quad x \geq -1+\sqrt{5} \\ x < -1 \end{cases}$$



$$\begin{cases} f'(x) \geq 0 \iff x \leq -1-\sqrt{5} \\ x < -1 \end{cases}$$



$-1-\sqrt{5}$ relative maximum

$-1+\sqrt{5}$ " "

$$f(-1-\sqrt{5}) = \arctan \frac{\sqrt{5}}{(-1-\sqrt{5})^2+4}$$

$$\wedge$$

$$f(-1+\sqrt{5}) = \arctan \frac{\sqrt{5}}{(-1+\sqrt{5})^2+4}$$

$-1+\sqrt{5}$ is an absolute maximum.

$$\lim_{x\to -1^+} f'(x) = \lim_{x\to -1^+} \frac{-x^2-2x+4}{(x^2+2)^2+(x+1)^2} =$$

$$= \frac{7}{9} = f'_+(-1)$$

$$\lim_{x\to -1^-} f'(x) = \lim_{x\to -1^-} \frac{x^2+2x-4}{(x^2+2)^2+(x+1)^2} =$$

$$= -\frac{7}{9}$$

$$\boxed{\lim_{x \to 0} \frac{(4\cos x - a)^2 - 4x^4}{x^4 \sin^2 x}}$$

Find the complex solution of

(*) $i z^2 + (1+2i) z + 1 = 0$

$z = x + iy$

$i(x^2 - y^2 + 2ixy) + (1+2i)(x+iy) + 1 = 0$

$$\begin{cases} -2xy + x - 2y + 1 = 0 \\ x^2 - y^2 + 2x + y = 0 \end{cases}$$

Try the direct formula for (*)

$$\frac{-(1+2i) + \text{sq. roots of} \left((1+2i)^2 - 4i\right)}{2i} =$$

$$(1+2i)^2 - 4i = 1 - 4 + \cancel{4i} - \cancel{4i}$$

$$= -3$$

$$= \frac{-(1+2i) \pm \sqrt{3}\, i}{2i} =$$

$$= \frac{-\cancel{2i} \pm \sqrt{3}\, \cancel{i}}{2\cancel{i}} - \frac{1}{2i} =$$

$$\boxed{\frac{-1 \pm \sqrt{3}}{2} + \frac{i}{2}}$$