Geometria 1 - mod. A - Lezione 36

Note Title

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -3 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \in M_3(\mathbb{R}) \qquad p_A(x) = -\det \begin{pmatrix} 1-x & 1 & 2 \\ -3 & -1-x & 0 \\ 1 & -1 & -x \end{pmatrix} =$$

$$= -\left[2(3+1+x) - x(x^2-1+3)\right] =$$

$$= -\left(8 + \cancel{2x} - x^3 + \cancel{x} - \cancel{3x}\right) \Rightarrow \boxed{p_A(x) = x^3 - 8}$$

$$p_A(x) = (x-2)(\underbrace{x^2+2x+4}_{\text{non ha radici in } \mathbb{R}})$$

$$\mathbb{R}^3 = \underset{\langle v_1 \rangle}{\underset{\shortparallel}{\mathrm{Ker}(A - 2\mathbb{1}_3)}} \oplus \underset{\langle v_2, v_3 \rangle}{\underset{\shortparallel}{\mathrm{Ker}(A^2 + 2A + 4\mathbb{1}_3)}}$$ per Lemma di decomp

Scelgo $\mathcal{V} = \{v_1, v_2, v_3\}$ allora $\alpha_{\mathcal{V}\mathcal{V}}(f_A) = \begin{pmatrix} \boxed{\cdot} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{}} \\ 0 & & \end{pmatrix}$

$\mathrm{Ker}(A - 2\mathbb{1}_3)$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ -3 & -3 & 0 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right) \underset{\substack{II+I \\ -\frac{1}{3}II}}{\overset{\substack{-I \\ eq}}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\overset{II-I}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & \cancel{2}\,1 & \cancel{2}\,1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} x_1 = -a + 2a \\ x_2 = -a \\ x_3 = a \end{array} \qquad \begin{pmatrix} a \\ -a \\ a \end{pmatrix}$$

$$\mathrm{Ker}(A - 2\mathbb{1}) = \langle \underset{= v_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \rangle$$

$$B = A^2 + 2A + 4\mathbb{1} = \begin{pmatrix} 0 & -2 & 2 \\ 0 & -2 & -6 \\ 4 & 2 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 & 2 \\ -3 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 6 \\ -6 & 0 & -6 \\ 6 & 0 & 6 \end{pmatrix}$$

$\mathrm{Ker}\, B$ $\quad (1\ 0\ 1 \mid 0)$ $\qquad \begin{array}{l} x_1 = -a \\ x_2 = b \\ x_3 = a \end{array} \quad \begin{pmatrix} -a \\ b \\ a \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$= \langle \underset{= v_2}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}, \underset{= v_3}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \rangle$$

$$\alpha_{\mathcal{V}\mathcal{V}}(f_A) = \begin{pmatrix} \boxed{2} & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & -1 \end{pmatrix}$$

$f_A(v_1) = Av_1 = 2v_1$

$f_A(v_2) = -v_2 - 3v_3$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -3 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \qquad Av_2 = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} = -\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_2} - 3\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3}$$

$$Av_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_2} - \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3}$$

$$A = \begin{pmatrix} -2 & 0 & -2 \\ 2 & 0 & 2 \\ 2 & 0 & 2 \end{pmatrix} \qquad p_A(x) = -\begin{vmatrix} -2-x & 0 & -2 \\ 2 & -x & 2 \\ 2 & 0 & 2-x \end{vmatrix} = +x \begin{vmatrix} -2-x & -2 \\ 2 & 2-x \end{vmatrix}$$

$$= x(x^2 - \cancel{4} + \cancel{4}) = x^3$$

Dunque $A^3 = \mathbb{O}$ unico autovalore di A è 0

$\operatorname{Ker} A$: risolvo $\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$ $\quad x_1 + x_3 = 0 \quad$ $1\ 0\ 1 \mid 0$ $\quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

dim 2 perché $\operatorname{rg} A = 1$

$$\operatorname{Ker} A = \left\langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v_2} \right\rangle \not\subseteq \operatorname{Ker} A^2 \underset{=}{\subseteq} \operatorname{Ker} A^3 = \mathbb{R}^3$$

Sia v t.c. $Av = 0$ $\qquad A^2 v = A(Av) = A(0) = 0$

$A^2 = \mathbb{O}$ Posso considerare un vettore v_3 di $\mathbb{R}^3$ t.c.

$\mathcal{V} = \{v_1, v_2, v_3\}$ sia base di $\mathbb{R}^3$

Ad $v_3 = e_3$ $\quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Matrice di f_A risp. alla base $\mathcal{V}$.

$f_A(v_1) = 0 \qquad f_A(v_2) = 0 \qquad f_A(v_3) = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = -2v_1 + 2v_2 \in \operatorname{Ker} A$

$$\alpha_{\mathcal{V}\mathcal{V}}(f_A) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{w_3 = e_3, \quad w_2 = f_A(e_3) = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}, \quad w_1 = e_2 \in \operatorname{Ker} A}_{\text{sono base di } \mathbb{R}^3}$$

$\operatorname{Ker} A^2$?

$\mathcal{W} = \{w_1, w_2, w_3\}$

$$\alpha_{ww}(f_A) = \begin{pmatrix} \boxed{0} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Esempio $\quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad p_A(x) := -\det \begin{pmatrix} -x & 1 & 0 \\ 1 & -x & 1 \\ 0 & -1 & -x \end{pmatrix}$

$$= -\left[-x(x^2+1) - 1(-x)\right] = -(-x^3 - x + x) = x^3$$

$$0 < \ker A \leq \ker A^2 \leq \ker A^3 = \mathbb{R}^3$$

dim 1 $\qquad$ dim 2

$$A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\ker A = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = \langle e_1 - e_3 \rangle \leq \ker A^2 = \langle e_1 - e_3, e_2 \rangle < \mathbb{R}^3$$

Se ora prendo $V = \langle e_1 - e_3, e_2, v_3 \rangle$ con $v_3 \in \mathbb{R}^3 \setminus \ker A^2$ a piacere

in generale non ottengo $\alpha_{vv}(f_A)$ "bello" ma ...

$w_3 \in \mathbb{R}^3 \setminus \ker A^2$ non deve ~~soddisfare~~ $x_1 + x_3 = 0$

$\overset{\shortparallel}{e_1} \qquad w_2 = Aw_3 = Ae_1 = e_2 \in \ker A \qquad w_1 = Aw_2 = Ae_2 = e_1 - e_3$

$$W = \{w_1, w_2, w_3\} \qquad \alpha_{ww}(f_A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = J_3$$

Se avessi scelto $w_3' = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad w_2' = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = Aw_3' \quad w_1' = Aw_2' = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \in \ker A$

$$\alpha_{w'w'}(f_A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = J_3$$

Perché funziona? $\qquad A^n = 0 \quad A \in M_n(\mathbb{R})$

Domani Merc. 11/1 lezione
8:30 – 10:15

Endomorfismi nilpotenti:

$\varphi : V \longrightarrow V$ nilpotente. $\dim V = n$, $\varphi^n = 0$ endom.

$P_\varphi(x) = x^n$, 0 unico autov.

φ triangolarizzabile

$$0 < \underset{\substack{\uparrow \\ \text{autosp.} \\ \text{relativo} \\ \text{a } \lambda=0}}{\operatorname{Ker}\varphi} < \operatorname{Ker}\varphi^2 < \operatorname{Ker}\varphi^3 \ \dots \ < \operatorname{Ker}\varphi^m = \dots = \operatorname{Ker}\varphi^n = V$$

Spiego: • $\boxed{\operatorname{Ker}\varphi^r \subseteq \operatorname{Ker}\varphi^{r+1}}$

Infatti: se $v \in \operatorname{Ker}\varphi^r \iff \varphi^r(v) = 0 \in V$

$$\Rightarrow \underbrace{\varphi(\varphi^r(v))}_{\varphi^{r+1}(v)} = \varphi(0) = 0 \Rightarrow v \in \operatorname{Ker}\varphi^{r+1}$$

• $\boxed{\text{Se } \operatorname{Ker}\varphi^r = \operatorname{Ker}\varphi^{r+1} \Rightarrow \operatorname{Ker}\varphi^{r+1} = \operatorname{Ker}\varphi^{r+2}}$

So che $\operatorname{Ker}\varphi^{r+1} \subseteq \operatorname{Ker}\varphi^{r+2}$ sempre.

Sia $v \in \operatorname{Ker}\varphi^{r+2}$. Allora $\underbrace{\varphi^{r+1}(\varphi v)}_{\varphi^{r+2}(v)} = 0$

$\Rightarrow \varphi(v) \in \operatorname{Ker}\varphi^{r+1} \overset{\text{ip.}}{=} \operatorname{Ker}\varphi^r \Rightarrow \varphi^r(\varphi(v)) = 0 \Rightarrow \varphi^{r+1}(v) = 0$

$\Rightarrow v \in \operatorname{Ker}\varphi^{r+1}$. Dunque ho anche $\supseteq$

$$\begin{tikzcd}
V \arrow[r, "\varphi"] & V \\
\operatorname{Ker}\varphi^{r+1} \arrow[u, phantom, "\cup"] \arrow[dr, "\varphi"] & \operatorname{Ker}\varphi^{r+1} \arrow[u, phantom, "\cup"] \\
\operatorname{Ker}\varphi^{r} \arrow[u, phantom, "\cup"] \arrow[dr] & \operatorname{Ker}\varphi^{r} \arrow[u, phantom, "\cup"] \\
\vdots \arrow[u, phantom, "\cup"] & \vdots \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

nilpotent

$$\varphi(\operatorname{Ker}\varphi^{r+1}) \subseteq \operatorname{Ker}\varphi^r$$

Sia $w \in \operatorname{Ker}\varphi^{r+1} \Rightarrow \varphi^{r+1}(w) = 0$

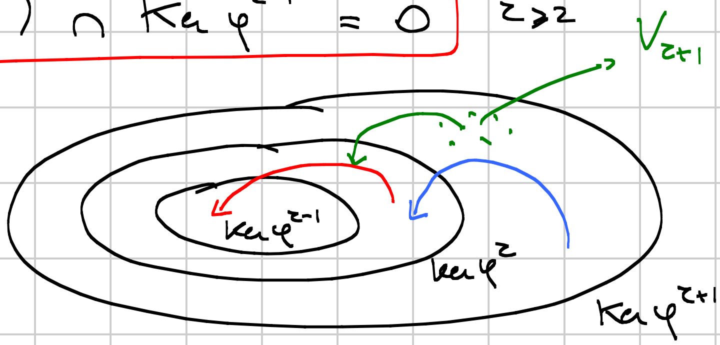
$\varphi^r(\varphi(w)) = \varphi^{r+1}(w) = 0 \Rightarrow \varphi(w) \in \operatorname{Ker}\varphi^r$

Notazione $V \xrightarrow{\varphi} V$ nilpotente

$$\operatorname{Ker}\varphi^{r+1} = \underset{\substack{\nwarrow \\ \text{complementare fissato}}}{V_{r+1}} \oplus \operatorname{Ker}\varphi^r$$

Osservo che $\varphi(V_{r+1}) \subseteq \underset{\operatorname{Ker}\varphi^{r-1}}{\underset{\cup\!\!\!\!\!\!\;\;\;\;}{\operatorname{Ker}\varphi^r}}$ (perché $\varphi(\operatorname{Ker}\varphi^{r+1}) \subseteq \operatorname{Ker}\varphi^r$)

$$\boxed{\varphi(V_{r+1}) \cap \operatorname{Ker}\varphi^{r-1} = 0} \qquad r \geq 2$$



" leggeteli come sp. vett. "

Infatti $\operatorname{Ker}\varphi^{r+1} = V_{r+1} \oplus \operatorname{Ker}\varphi^r \supseteq V_{r+1} \oplus \operatorname{Ker}\varphi^{r-1} \oplus \underbrace{\cdots}_{\varphi(V_{r+1})}$

Sia $v \in \operatorname{Ker}\varphi^{r-1} \cap \varphi(V_{r+1})$ $\quad v = \varphi(w)$ con $w \in V_{r+1}$

e $\varphi^{r-1}(v) = 0$. Ossia $\varphi^{r-1}(\varphi(w)) = 0$ ossia

$\varphi^r(w) = 0 \Rightarrow w \in \operatorname{Ker}\varphi^r$

Dunque $w \in \operatorname{Ker}\varphi^r \cap V_{r+1} = 0 \Rightarrow w = 0 \Rightarrow v = 0$

$\operatorname{Ker}\varphi^{r+1} = V_{r+1} \oplus \operatorname{Ker}\varphi^r$

$\varphi|_{V_{r+1}}$ è iniettiva ossia vettori $v_1, \dots v_s$ in V_{r+1} sono l. ind. $\Leftrightarrow \varphi(v_1), \dots, \varphi(v_s)$ sono l. ind.

$\Rightarrow$)

Siano $v_1, \dots v_s$ in V_{r+1} linearmente indipendenti e

sia $\sum \alpha_i \varphi(v_i) = 0$; devo mostrare che $\alpha_i = 0$

Da $\underset{\text{lineare}}{\sum \alpha_i \varphi(v_i) = 0}$, segue che $\varphi(\sum \alpha_i v_i) = 0$

Dunque ho che $\sum \alpha_i v_i \in \underset{\operatorname{Ker}\varphi^r}{\underset{\cap}{\operatorname{Ker}\varphi}}$ e $\sum \alpha_i v_i \in V_{r+1}$

$\Rightarrow \sum \alpha_i v_i \in \operatorname{Ker}\varphi^r \cap V_{r+1} = 0 \Rightarrow \sum \alpha_i v_i = 0$

$\Rightarrow \alpha_i = 0 \; \forall i$ perché i v_i sono l. indipendenti.

(Viceversa esercizio)

Ora decomponiamo V come segue.

$\varphi : V \longrightarrow V$ nilpot. $\varphi^m = 0$ x^m è pol. minimo

$$0 < \operatorname{Ker}\varphi < \operatorname{Ker}\varphi^2 < \dots < \operatorname{Ker}\varphi^m = \operatorname{Ker}\varphi^{m+1} = \dots = \operatorname{Ker}\varphi^n$$

$\operatorname{Ker}\varphi^m = V$ $n = \dim V$

$$V = \operatorname{Ker}\varphi^m = V_m \oplus \underbrace{\operatorname{Ker}\varphi^{m-1}}_{\underbrace{\varphi(V_m) \oplus V'_{m-1}}_{V_{m-1}} \oplus \operatorname{Ker}\varphi^{m-2}}$$

$$= V_m \oplus V_{m-1} \oplus \underbrace{\operatorname{Ker}\varphi^{m-2}}_{\underbrace{V'_{m-2} \oplus \varphi V_{m-1}}_{V_{m-2}} \oplus \operatorname{Ker}\varphi^{m-3}}$$

$$\varphi V_{m-1} = \varphi^2 V_m \oplus \varphi V'_{m-1}$$

$$= V_m \oplus V_{m-1} \oplus \dots V_3 \oplus \underbrace{V_2 \oplus \underbrace{\operatorname{Ker}\varphi}_{V_1}}_{\operatorname{Ker}\varphi^2}$$

with $V_3 \oplus V_2 \oplus \operatorname{Ker}\varphi$ $= \operatorname{Ker}\varphi^3$

$$\operatorname{Ker}\varphi^j = \bigoplus_{i=1}^{j} V_i \qquad V = \operatorname{Ker}\varphi^m = \bigoplus_{i=1}^{m} V_i$$

Scrivo una base di V formata unendo basi dei V_i

$$\operatorname{Ker}\varphi^m = V = V_m \oplus \operatorname{Ker}\varphi^{m-1} = V_m \oplus V_{m-1} \oplus \operatorname{Ker}\varphi^{m-2} = \dots$$

V_m $\mathcal{V}_m$ base di V_m

V_{m-1} $\varphi(\mathcal{V}_m) \cup \mathcal{V}'_{m-1}$ base di V_{m-1} con $\mathcal{V}'_{m-1}$ base di V'_{m-1}

V_{m-2} $\varphi^2(\mathcal{V}_m) \cup \varphi(\mathcal{V}'_{m-1}) \cup \mathcal{V}'_{m-2}$ base di V_{m-2} con $\mathcal{V}'_{m-2}$ .. V'_{m-2}

⋮

NB se $V'_{m-1} = 0$ allora $\mathcal{V}'_{m-1}$ non compare

V_m $\{v_{m,1}, \dots v_{m,s_m}\} = \mathcal{V}_m$

V_{m-1} $\{\varphi(v_{m,1}), \dots, \varphi(v_{m,s_m}), v_{m-1,1}, \dots v_{m-1,s_{m-1}}\}$ base di $\mathcal{V}'_{m-1}$

V_{m-2} $\{\varphi^2(v_{m,1}), \dots, \varphi^2(v_{m,s_m}), \varphi(v_{m-1,1}) \dots \varphi(v_{m-1,s_{m-1}}), v_{m-2,1} \dots v_{m-2,s_{m-2}}\}$

$s_m = \dim V_m$

$s_{m-1} = \dots V_{m-1}$.

Immagino di scrivere in colonna questi vettori:

$$
\begin{array}{cccccc}
v_{m,1} & \dots & v_{m,s_m} & & & \\
\varphi(v_{m,1}) & & \varphi(v_{m,s_m}) & v_{m-1,1} & \dots & \\
\vdots & & & \varphi(v_{m-1,1}) & & v_{m-2,1} \\
\vdots & & & & & \vdots \\
\varphi^m(v_{m,1}) & & \varphi^m(v_{m,s_m}) & \varphi^{m-1}(v_{m-1,1}) & & \varphi^{m-2}(v_{\dots}
\end{array}
$$

Scelgo la base di V scrivendo i vettori della tabella sopra dal basso in alto da sinistra a destra

Ogni colonna dà un blocco di Jordan di ordine pari all'altezza della colonna.