$$\int_e^{e^3} \frac{1}{x\left((\log x)^2 + \log(x^3)\right)}\,dx =$$

$$= \int_e^{e^3} \frac{1}{x\left((\log x)^2 + 3\log x\right)}\,dx =$$

$y = \log(x)$

$$= \int_1^3 \frac{1}{(y^2+3y)}\,dy$$

$dy = \frac{1}{x}dx$

$$y^2+3y = y(y+3)$$

$$\frac{A}{y} + \frac{B}{y+3} = \frac{1}{y^2+3y}$$

$$A(y+3) + By = 1$$

$$\begin{cases} A+B = 0 \\ 3A = 1 \end{cases} \qquad \begin{cases} B = -\frac{1}{3} \\ A = \frac{1}{3} \end{cases}$$

$$\int_1^3 \left(\frac{1}{3y} - \frac{1}{3(y+1)}\right) dy =$$

$$\frac{1}{3}\log y - \frac{1}{3}\log(y+1)\Big|_1^3 =$$

$$= \log \frac{\sqrt[3]{y}}{\sqrt[3]{y+1}}\Big|_1^3 =$$

$$\log\frac{\sqrt[3]{3}}{\sqrt[3]{4}} - \log\frac{1}{\sqrt[3]{2}}$$

$$\int \frac{x^4+2x-3}{x^3+2x^2+x}\, dx =$$

$$\begin{array}{l|l}
x^4+0x^3+0x^2+2x-3 & x^3+2x^2+x \\
\hline
x^4+2x^3+x^2 & x-2 \\
\hline
-2x^3-x^2+2x-3 & \\
-2x^3-4x^2-2x & \\
\hline
3x^2+4x-3 &
\end{array}$$

$$= \int \left((x-2) + \frac{3x^2+4x-3}{x^3+2x^2+x} \right) dx$$

$$= \frac{(x-2)^2}{2} + \int$$

$$x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$$

$$\frac{3x^2+4x-3}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} -$$

$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x^3+2x^2+x}$$

$$= \frac{Ax^2+2Ax+A+Bx^2+Bx+Cx}{x^3+2x^2+x}$$

$$\begin{cases} 3 = A + B \\ 4 = 2A + B + C \\ -3 = A \end{cases}$$

$$\begin{cases} B = 3 - A = 6 \\ C = 4 - 2A - B = 4+6-6 = 4 \end{cases}$$

$$\int \ldots = \frac{(x-2)^2}{2} + \int -\frac{3}{x} + \frac{6}{x+1} + \frac{4}{(x+1)^2}$$

$$= \frac{(x-2)^2}{2} - 3\log|x| + 6\log|x+1| - \frac{4}{(x+1)} + C$$

$$C \in \mathbb{R}$$

$$\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx =$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\int \left(\frac{1}{4} - \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right) dx =$$

$$\frac{1}{4}x - \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1+\cos 4x}{2} dx + C$$

$$= \frac{1}{4}x - \frac{\sin 2x}{4} + \frac{1}{8}x + \frac{1}{8}\sin 4x \cdot \frac{1}{4} + C$$

$$\int_2^{+\infty} \frac{1}{x^2 (\log x)^\alpha}\, dx$$ Study convergence for $\alpha \in \mathbb{R}$

$\alpha \geq 0$ $$\frac{1}{x^2(\log x)^\alpha} \leq \frac{1}{x^2}$$

Converges for $\alpha \geq 0$

$\alpha < 0$ $$\frac{1}{x^2(\log x)^\alpha} = \frac{(\log(x))^{-\alpha}}{x^2}$$

$$\lim_{x \to +\infty} \frac{\frac{(\log(x))^{-\alpha}}{x^2}}{\frac{1}{x^{\frac{3}{2}}}} =$$

$$\lim_{x \to +\infty} \frac{(\log(x))^{-\alpha}}{x^{\frac{1}{2}}} =$$

$y = \log x$

$x = e^y$

$$\lim_{y \to +\infty} \frac{(y)^{-\alpha}}{e^{y/2}} = 0$$

$\Rightarrow$ convergence $\forall \alpha \in \mathbb{R}$

$$\int_1^{+\infty} \frac{1}{x^2(\log x)^\alpha}\,dx = \int_1^2 \frac{1}{x^2(\log x)^\alpha} + \boxed{+\int_2^{+\infty} \frac{1}{x^2(\log x)^\alpha}}$$

converges $\forall \alpha \in \mathbb{R}$

Study

$$\int_1^2 \frac{1}{x^2(\log x)^\alpha}\,dx \quad \text{converges}$$

$$\Updownarrow$$

$$\int_1^2 \frac{1}{(\log x)^\alpha}\,dx$$

indeed

$$\frac{1}{4}\frac{1}{(\log x)^\alpha} < \frac{1}{x^2(\log x)^\alpha} \leq \frac{1}{(\log x)^\alpha}$$

$$\lim_{x\to 1^+} \frac{\frac{1}{x^2(\log x)^\alpha}}{\frac{1}{(\log x)^\alpha}} = 1$$

$$\int_1^2 \frac{1}{(\log x)^\alpha}\,dx = \lim_{k\to 1^+} \int_k^2 \frac{1}{(\log)^\alpha}\,dx$$

$y = \log x$

$dy = \frac{1}{x}\,dx$

$dx = x\,dy = e^y\,dy$

$$\lim_{k\to 1} \int_{\log k}^{\log 2} \frac{e^y}{y^\alpha}\,dy = \boxed{\int_0^{\log 2} \frac{e^y}{y^\alpha}\,dy}$$

converges

$\frac{e^y}{y^\alpha} \sim \frac{1}{y^\alpha}$ if and only if

$\alpha < 1$

The whole convergence of $\int_1^{+\infty} \frac{1}{x^2 (\log x)^\alpha}$ is true for $\boxed{\alpha < 1}$

$$\boxed{\int_{-\infty}^{+\infty} \frac{1}{x^2+9}\,dx}$$

it converges, by definition, if and only if $\int_0^{+\infty} \frac{1}{x^2+9}\,dx$ and $\int_{-\infty}^{0} \frac{1}{x^2+9}\,dx$ converge

$$\int_0^{+\infty} \frac{1}{x^2+4} := \lim_{k\to+\infty} \int_0^k \frac{1}{x^2+4}\,dx =$$

$$\lim_{k\to+\infty} \frac{1}{4} \int_0^k \frac{1}{\left(\frac{x^2}{4}+1\right)}\,dx \overset{y=\frac{x}{2}}{=}$$

$$= \lim_{k\to+\infty} \frac{1}{4} \int_0^{\frac{k}{2}} \frac{2\,dy}{(y^2+1)} =$$

$$= \lim_{k\to+\infty} \frac{1}{2} \operatorname{arctg} y \Big|_0^{\frac{k}{2}} =$$

$$= \lim_{k\to+\infty} \frac{1}{2} \underbrace{\left(\operatorname{arctg} \frac{k}{2}\right)}_{\to\frac{\pi}{2}} = \frac{\pi}{4}$$

Complete the exercise:

$$\boxed{\int_0^k \frac{1}{y^\alpha}\,dy \text{ convergent} \iff \alpha < 1} = (*)$$

$k > 0$

$$\int_k^{+\infty} \frac{1}{y^\alpha}\,dy \text{ convergent} \iff \alpha > 1$$

$$(*) = \lim_{c \to 0^+} \int_c^k \frac{1}{y^\alpha}\,dy =$$

if $\boxed{\alpha = 1}$ $= \lim_{c \to 0^+} \log y \Big|_c^k =$

$$\lim_{c \to 0} \left(\log k - \log c \right) = +\infty$$

if $\boxed{\alpha \neq 1}$

$$\lim_{c \to 0} \frac{y^{-\alpha+1}}{-\alpha+1} \Bigg|_c^k = \begin{cases} \boxed{k^{-\alpha+1}} & \text{for } -\alpha+1 > 0 \iff \alpha < 1 \\ \boxed{+\infty} & \text{for } -\alpha+1 < 0 \quad \alpha > 1 \end{cases}$$

for any $a, b \in \mathbb{R}$

$a < b$

prove that

$$\int_a^b \frac{1}{\sqrt{(x-b)(a-x)}}\,dx = \pi$$

---

observe that if

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$$

and if $a=-1$ $b=1$

$$\int_{-1}^{1} \frac{1}{\sqrt{(x-1)(-1-x)}} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}$$

Idea : $\boxed{y(x) = Ax + B}$

s.t.

$$y(a) = -1 \quad \Longleftrightarrow \quad \begin{cases} Aa + B = -1 \\ Ab + B = 1 \end{cases}$$
$$y(b) = 1$$

finish it.

$$\sum \frac{\left(x - \sin\frac{1}{x}\right)^n}{1 + n(x+4)^2}$$

$x \geq 0$
a parameter

Study convergence
for every $x \geq 0$.