ES $\dim_{\mathbb{Q}} V = 5 \qquad \gamma = \{v_1, \dots, v_5\}$

$$\alpha_{\gamma\gamma}(\varphi) = A = \begin{pmatrix} -7 & 0 & -3 & -12 & 0 \\ 0 & 4 & -1 & -1 & 2 \\ -6 & 0 & 0 & -8 & 0 \\ 6 & 0 & 3 & 11 & 0 \\ 0 & -5 & 2 & 2 & -3 \end{pmatrix}$$

$$P_\varphi(x) = \begin{vmatrix} x+7 & 0 & 3 & 12 & 0 \\ 0 & x-4 & 1 & 1 & -2 \\ 6 & 0 & x & 8 & 0 \\ -6 & 0 & -3 & x-11 & 0 \\ 0 & 5 & -2 & -2 & x+3 \end{vmatrix} =$$

$$= \begin{vmatrix} x+7 & 12 & 3 & 0 & 0 \\ -6 & x-11 & -3 & 0 & 0 \\ 6 & 8 & x & 0 & 0 \\ 0 & 1 & 1 & x-4 & -2 \\ 0 & -2 & -2 & 5 & x+3 \end{vmatrix} = \begin{vmatrix} x+7 & 12 & 3 \\ -6 & x-11 & -3 \\ 6 & 8 & x \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} x-4 & -2 \\ 5 & x+3 \end{vmatrix}$$

$$\underbrace{\qquad}_{=} \quad x^2 - x - 12 + 10 =$$

$$= x^2 - x - 2 = (x+1)(x-2)$$

$$\begin{vmatrix} x+7 & 12 & 3 \\ -6 & x-11 & -3 \\ \dots \end{vmatrix} = (x+7)\begin{vmatrix} x-11 & -3 \\ \dots & x \end{vmatrix} + 6 \begin{vmatrix} 12 & 3 \\ \dots & x \end{vmatrix}$$

$$\begin{vmatrix} 6 & 8 & x \end{vmatrix} - (x+7) \Big| \quad 8 \quad x \quad \Big( 8 \quad x \Big)$$

$$+6 \begin{vmatrix} 12 & 3 \\ x-11 & -3 \end{vmatrix}$$

$$= (x+7)(x^2 - 11x + 24) + 6(12x - 24) + 6(-3x-3) =$$

$$= \_\_\_ = (x-2)(x+1)(x-3)$$

$$P_\varphi(x) = (x-2)^2 (x+1)^2 (x-3)$$

| | | | |
|---|---|---|---|
| A. VAL | 2 | $-1$ | 3 |
| $m_a$ : | 2 | 2 | 1 |
| $m_g$ : | 2 | 2 | 1 |

$-1$ $\quad A - (-\mathbb{1}) = \begin{pmatrix} -6 & 0 & -3 & -12 & 0 \\ 0 & 5 & -1 & -1 & 2 \\ -6 & 0 & 1 & -8 & 0 \\ 6 & 0 & 3 & 12 & 0 \\ 0 & -5 & 2 & 2 & -2 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -5 \end{pmatrix} \in \operatorname{Ker}(A + \mathbb{1})$$

$$\begin{matrix} \mathrm{I}/-3 \\ \mathrm{II} \end{matrix} \quad \begin{pmatrix} 2 & 0 & 1 & 4 & 0 \\ 0 & 5 & -1 & -1 & 2 \end{pmatrix} \quad \begin{matrix} \mathrm{I} \\ \mathrm{II} + \mathrm{III}/\ldots \end{matrix}$$

$$A+\mathbb{1} \quad \begin{matrix} \text{III}-\text{I} \\ \text{IV}+\text{I} \\ \text{V}+\text{II} \end{matrix} \begin{pmatrix} 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{matrix} \text{III}/4 \\ \text{IV} \\ \text{V}-\text{III}/4 \end{matrix}$$

$$\begin{pmatrix} 2 & 0 & 1 & 4 & 0 \\ 0 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \;\text{I}-\text{III}\; \begin{pmatrix} 2 & 0 & 0 & 3 & 0 \\ 0 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ \hline \\ \hline \end{pmatrix}$$

$$\text{Ker} = \left\langle \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \\ -2 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$$

$$\begin{cases} 2a+3d=0 \\ 5b+2e=0 \\ c+d=0 \end{cases}$$

$$A-2\mathbb{1}_5 = \begin{pmatrix} -9 & 0 & -3 & -12 & 0 \\ 0 & 2 & -1 & -1 & 2 \\ -6 & 0 & -2 & -8 & 0 \\ 6 & 0 & 3 & 9 & 0 \\ 0 & -5 & 2 & 2 & -5 \end{pmatrix} \sim$$

$$\text{Ker}(A-2\mathbb{1}) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

3 $\quad A - 3\mathbb{1} = \begin{pmatrix} -10 & 0 & -3 & -12 & 0 \\ 0 & 1 & -1 & -1 & 2 \\ -6 & 0 & -3 & -8 & 0 \\ 6 & 0 & 3 & 8 & 0 \\ 0 & -5 & 2 & 2 & -6 \end{pmatrix}$ $\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \sim \mathrm{III}+\mathrm{IV} \\ 10\,\mathrm{III}-6\,\mathrm{I} \\ 5\,\mathrm{II}+\mathrm{V} \end{array}$

$$= \begin{pmatrix} 10 & 0 & 3 & 12 & 0 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -8 & 0 \\ 0 & 0 & -3 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 3 & 12 & 0 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$$

$\mathrm{V} + \frac{\mathrm{IV}}{-4}$

$$\mathrm{Ker} = \left\langle \begin{pmatrix} \\ \\ \\ 4 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{cases} 10a + 3c + 12d = 0 \\ b - c - d + 2e = 0 \\ 3c + 2d = 0 \\ -d + 4e = 0 \end{cases}$$

$\boxed{e = 1} \Rightarrow -d + 4 = 0 \Rightarrow \boxed{d = 4} \Rightarrow 3c + 8 = 0$

$\Rightarrow \boxed{c = -\frac{8}{3}}$

$$10a - 3 \cdot \frac{8}{3} + 12 \cdot 4 = 0$$

$$10a - 8 + 48 = 0$$

$$\boxed{a = -4}$$

$$b + \frac{8}{3} - 4 + 2 = 0 \qquad b + \frac{8-12+6}{3} = 0$$

$$\boxed{b = -\frac{2}{3}}$$

$$\left\langle \begin{pmatrix} -4 \\ -2/3 \\ -8/3 \\ 4 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -12 \\ -2 \\ -8 \\ 12 \\ 3 \end{pmatrix} \right\rangle$$

$$D = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 3 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & +3 & 0 & 1 & 12 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 8 \\ 0 & -2 & 0 & -1 & -12 \\ -5 & 0 & -1 & 0 & -3 \end{pmatrix}$$

$\mathrm{Ker}(A - 3\mathbb{1})$

$\mathrm{Ker}(A + \mathbb{1})$ $\qquad$ $\mathrm{Ker}(A - 2\mathbb{1})$

$$A=\begin{pmatrix} -2 & 0 & -4 & 0 \\ -1 & -1 & -1 & -2 \\ 4 & 0 & 6 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$p_\varphi(x)=\begin{vmatrix} x+2 & 0 & 4 & 0 \\ 1 & x+1 & 1 & 2 \\ -4 & 0 & x-6 & 0 \\ -1 & -2 & -1 & x-3 \end{vmatrix} =$$

$$= -\begin{vmatrix} x+2 & 4 & 0 & 0 \\ 1 & 1 & x+1 & 2 \\ -4 & x-6 & 0 & 0 \\ -1 & -1 & -2 & x-3 \end{vmatrix} = \begin{vmatrix} x+2 & 4 & 0 & 0 \\ -4 & x-6 & 0 & 0 \\ 1 & 1 & x+1 & 2 \\ -1 & -1 & -2 & x-3 \end{vmatrix}$$

$$= \big((x+2)(x-6)+16\big)\big((x+1)(x-3)+4\big) =$$

$$= (x-2)^2(x-1)^2$$

$$\boxed{\begin{matrix} ma(2)=2 \\ ma(1)=2 \end{matrix}}$$

Autospazi:

$$\begin{pmatrix} -4 & 0 & -4 & 0 \\ -1 & -3 & -1 & -2 \end{pmatrix}$$

$$A - 2\mathbb{1} = \begin{pmatrix} -1 & 3 & 1 & 2 \\ 4 & 0 & 4 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \quad rk = 3$$

$$mg(2) = 1$$

$$\ker(\varphi - 2id) = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

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$\mathbb{R}^2$ : polinomio minimo di proiezione e simmetria

$$\pi \text{ proiezione} \Leftrightarrow \pi^2 = \pi$$

$$\sigma \text{ simmetria} \Leftrightarrow \sigma^2 = id$$

$$\varphi : \varphi^k + a_{k-1}\varphi^{k-1} + \dots + a_0 \Rightarrow$$

$$M\varphi(x) \mid x^k + a_{n-1}x^{k+1} + \dots + a_0$$

$\pi$ $\quad \pi^2 = \pi \Leftrightarrow \pi^2 - \pi = 0$

$$\Rightarrow M\pi(x) \mid x^2 - x = x(x-1)$$

$$M\pi(x) = x(x-1)$$

$\sigma$ $\quad \sigma^2 = id \Rightarrow \sigma^2 - id = 0$

$$\Rightarrow M\sigma(x) = x^2 - 1 = (x-1)(x+1)$$

dim $V = N$, $\varphi: V \to V$ con autovalori reali t.c. $\varphi^K = id_V$, $K \in \mathbb{N}$

Dimostrare che $\varphi$ è diagonalizzabile e che $\varphi^2 = id_V$

$\varphi^k = id_V \Rightarrow M_\varphi(x) \mid x^k - 1$

De Moivre: $x^k - 1$ ha radici tutte distinte

$\Rightarrow M_\varphi(x)$ ha radici distinte $\Rightarrow \varphi$ è diagonalizzabile (II criterio)

$\varphi$ autovalori reali $\Rightarrow 1, -1$

- $K$ pari: $M_\varphi(x) \mid x^2 - 1 = (x-1)(x+1)$

  $\Rightarrow \varphi^2 = id_V$

- $K$ dispari: $M_\varphi(x) \mid x - 1 \Rightarrow \varphi = id$

  $(\Rightarrow \varphi^2 = id_V)$

  $M_\varphi(x) = x - 1$

$V, W$ $\mathbb{C}$ $n, m$

$\varphi: V \to W$ lineari $\varphi \circ \psi$ hanno
$\psi: W \to V$ $\psi \circ \varphi$ gli stessi
a.vol. non nulli?
E addirittura con le stesse mg?

$n < m$ $\quad \varphi \circ \psi : W \to W$

$rk\, \varphi \circ \psi \leq n < m$
0 sarà a.vol. di $\varphi \circ \psi$

$v \in V$ sia a.vett per $\psi \circ \varphi$ rel.
ad. a.vol $c \neq 0$
$(\psi \circ \varphi)(v) = cv$

($P_{\psi \circ \varphi}$ ha sempre radici perché siamo in $\mathbb{C}$)

$(\varphi \circ \psi \circ \varphi)(v) = \varphi(\psi \circ \varphi)(v) = \varphi(cv) = c\,\varphi(v)$

$\varphi(v) \in W$ è a.vett. relativo ad
a.vol. $c \neq 0$ per $\varphi \circ \psi$

Analogamente se $w \in W$ è a.vett.
per $c \neq 0$ per $\varphi \circ \psi \Rightarrow \psi(w) \in V$ è

a. vett. per $c \neq 0$ per $\varphi \circ \psi$

$\varphi$ e $\psi$ definiscono un isomorfismo tra gli autospazi relativi allo stesso autovalore

di $\varphi \circ \psi$ e $\psi \circ \varphi$

$\Downarrow$

Gli a. spazi rel a $c$ per $\varphi \circ \psi$ e $\psi \circ \varphi$ sono isomorfi

$\Rightarrow$ stessa dimensione

$\Rightarrow$ stesse molt. geom. $\forall c$ a.val di $\varphi \circ \psi$ e $\psi \circ \varphi$