9.6. Exercises

Exercise 9.6.1 ($\star$). For any of the following series, compute the partial sums and discuss the convergence (computing the eventual sum) of the series:

$$1.\ \sum_{n=0}^{\infty}\frac{1}{(2n-1)(2n+1)}.\quad 2.\ \sum_{n=0}^{\infty}\left(\sqrt{n(n+1)}-\sqrt{n(n-1)}-1\right).\quad 3.\ \sum_{n=2}^{\infty}\log\left(1-\frac{1}{n^2}\right).$$

Exercise 9.6.2. For any of the following series $\sum_n a_n$ find b_n such that $a_n \leqslant b_n$ and $\sum_n b_n$ converges.

$$1.\ \sum_{n=1}^{\infty}\frac{1}{n^{3+(-1)^n}}.\quad 2.\ \sum_{n=0}^{\infty}\frac{1+\sin n}{2^n}.\quad 3.\ \sum_{n=0}^{\infty}2^{\sqrt{n}}3^{-n}.$$

Exercise 9.6.3. Applying the asymptotic comparison test, we discuss the convergence of the following series

$$1.\ \sum_{n=1}^{\infty}\frac{(n+2)^n}{n^{n+2}}.\qquad 2.\ \sum_{n=1}^{\infty}\frac{n+\log n}{(n-\log n)^3}.\quad 3.\ \sum_{n=0}^{\infty}(\sqrt[3]{n+1}-\sqrt[3]{n}).$$

$$4.\ \sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}.\qquad 5.\ \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n+2}{n+3}\right)^n.\quad 6.\ \sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-n}(\sqrt{n+1}-\sqrt{n})}.$$

$$7.\ \sum_{n=0}^{\infty}n^{\beta}\left(1-\sqrt{\frac{n^3}{n^3+1}}\right),\quad (\beta\in\mathbb{R}).$$

Exercise 9.6.4. Applying root or ratio tests, determine if the following series converge:

$$1.\ \sum_{n=0}^{\infty}5^{(-1)^n-n}.\qquad 2.\ \sum_{n=1}^{\infty}\frac{(n!)^2}{n^2(2n)!}.\qquad 3.\ \sum_{n=1}^{\infty}\frac{n^k}{n!}.\qquad 4.\ \sum_{n=0}^{\infty}\frac{n^{n+1}}{3^n(n+1)!}.$$

$$5.\ \sum_{n=1}^{\infty}\frac{n^{nx}}{n!},\ (x\in\mathbb{R}).\quad 6.\ \sum_{n=1}^{\infty}\frac{(n!)^x}{n^n},\ (x\in\mathbb{R}).\quad 7.\ \sum_{n=1}^{\infty}x^{n!},\ (x\geqslant 0).\quad 8.\ \sum_{n=0}^{\infty}n!\,x^n,\ (x\geqslant 0).$$

Exercise 9.6.5. Applying Leibniz test, determine if the following series converge:

$$1.\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}.\qquad 2.\ \sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n+1}-\sqrt{n}}{n}.\quad 3.\ \sum_{n=1}^{\infty}(-1)^n(\sqrt[n]{3}-1).$$

$$4.\ \sum_{n=0}^{\infty}(-1)^n\left(1-\cos\frac{1}{\sqrt{n+1}}\right).\quad 5.\ \sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}-1}{\cos(n\pi)}.\qquad 6.\ \sum_{n=1}^{\infty}(-1)^n\left(\sqrt{1+\frac{1}{n}}-1\right).$$

$$7.\ \sum_{n=1}^{\infty}(-1)^n\sin\frac{1}{n}.\qquad 8.\ \sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{n}\right)^n.\qquad 9.\ \sum_{n=1}^{\infty}(-1)^n\frac{n+1}{4^n}.$$

$$10.(\star)\ \sum_{n=1}^{\infty}(-1)^n\frac{(n!)^2}{n^2(2n)!}.\qquad 11.(\star)\ \sum_{n=1}^{\infty}(-1)^n\frac{\log n}{n}.$$

Exercise 9.6.2 (Comparison)

1. $\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$, $a_n = \frac{1}{n^{3+(-1)^n}} \leq \frac{1}{n^2} = b_n$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (harmonic with exponent 2)

2. $\sum_{n=1}^{\infty} \frac{1+\sin n}{2^n}$ $\quad a_n = \frac{1+\sin n}{2^n} \leq \frac{2}{2^n} = b_n$

$2 \cdot \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with ratio $\frac{1}{2} < 1$)

3. $\sum_{n=1}^{\infty} 2\sqrt{n}\, 3^{-n}$ $\quad a_n = \frac{2\sqrt{n}}{3^n} \leq \frac{2 \cdot 2^n}{3^n} = \quad = 2 \cdot \left(\frac{2}{3}\right)^n = b_n$

$2\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges (geometric with ratio $\frac{2}{3} < 1$)

Exercise 9.6.3 (asymtotic comparison)

$$a_n = \frac{(n+2)^2}{n^{n+2}}$$

1. $\sum_{n=1}^{\infty} \frac{(n+2)^n}{n^{n+2}} = \sum_{n=1}^{\infty} \frac{n^n\left(1+\frac{2}{n}\right)^n}{n^{n+2}} =$

$= \sum_{n=1}^{\infty} \frac{\left(1+\frac{2}{n}\right)^n}{n^2}$ From $\lim_{n\to 0} \frac{\frac{\left(1+\frac{2}{n}\right)^n}{n^2}}{\underbrace{\frac{e^2}{n^2}}_{b_n}}$.

$$\lim_{n\to\infty} \frac{\left(\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right)^2}{e^2} = 1$$ the sequence a_n is asymtotic to $b_n = \frac{e^2}{n^2}$

so by asymt. comparison it converges because $\sum_{n=1}^{\infty} b_n$ converges

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2) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\log n}{(n-\log n)^3}$

$$a_n = \frac{n\left(1+\frac{\log n}{n}\right)}{n^3\left(1-\frac{\log n}{n}\right)^3} \sim \frac{1}{n^2} =: b_n$$

By asymt. comp. the series converges because $\sum_{n=1}^{\infty} b_n$ converges

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3) $\displaystyle\sum_{n=1}^{\infty} \sqrt[3]{n+1} - \sqrt[3]{n}$

$$a_n = \sqrt[3]{n+1} - \sqrt[3]{n} = \sqrt[3]{n}\left(1+\frac{1}{n}\right)^{1/3} - \sqrt[3]{n} =$$

$$= \sqrt[3]{n}\left(1+\frac{1}{3n}+o\left(\frac{1}{n}\right)\right) - \sqrt[3]{n} =$$

$$= \frac{1}{3n^{2/3}} + o\left(\frac{1}{n^{2/3}}\right) \sim \frac{1}{3n^{2/3}} = b_n$$

Hence $\sum a_n$ does not converge because $\sum b_n$ does not converge (harmonic with exponent <1)

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4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ $\quad a_n = \frac{1}{n\sqrt[n]{n}} \sim \frac{1}{n} =: b_n$,

because $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{\log n}{n}} = 1$

Hence $\sum a_n$ does not converge

because $\sum b_n$ " " " .

5) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n+2}{n+3}\right)^n$

$a_n = \frac{1}{n}\left(\frac{n+3}{n+3} - \frac{1}{n+3}\right)^n = \frac{1}{n}\left(\left(1-\frac{1}{n+3}\right)^{n+3}\right)^{\frac{n}{n+3}}$

Since $\lim_{n\to+\infty}\left(\left(1-\frac{1}{n+3}\right)^{n+3}\right)^{\frac{n}{n+3}} = \frac{1}{e}$

$\Longrightarrow \quad a_n \sim \frac{1}{n} = b_n$

so $\sum_{n=1}^{\infty} a_n$ does not converge

(because $\sum_{n=1}^{\infty} b_n$ does not converge)

6) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-n}\left(\sqrt{n+1}-\sqrt{n}\right)} = \sum_{n=1}^{+\infty} a_n$

with

$a_n = \frac{1}{n\sqrt{1-\frac{1}{n}}\,\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-1\right)} =$

$= \frac{1}{n^{\frac{3}{2}}\sqrt{1-\frac{1}{n}}\left(\cancel{1}+\left(\frac{1}{n}\right)^{\frac{1}{2}}+o\left(\left(\frac{1}{n}\right)^{\frac{1}{2}}\right)\cancel{-1}\right)} =$

$\sim \dfrac{1}{n^2} = b_n$

so $\sum_{n=1}^{\infty} a_n$ converges because $\sum_{n=1}^{\infty} b_n$ converges.

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7) $\sum_{n=0}^{+\infty} n^\beta \left(1 - \sqrt{\dfrac{n^3}{n^3+1}}\right)$ $\quad \beta \in \mathbb{R}$

$a_n = n^\beta \left(1 - \sqrt{\dfrac{n^3}{n^3+1}}\right) =$

$= n^\beta \left(1 - \sqrt{1 - \dfrac{1}{n^3+1}}\right) =$

$= n^\beta \left(1 - \left(1 - \dfrac{1}{2(n^3+1)}\right) + o\left(\dfrac{1}{n^3+1}\right)\right)$

$\sim \dfrac{n^\beta}{2(n^3+1)} \sim \dfrac{1}{n^{3-\beta}} = b_n$

So the series converges

if and only if $\sum_{n=1}^{\infty} b_n$ converges,

that is, if and only if $3-\beta > 1$

$\iff \beta < 2$.

Esercizio 6.9.4

root or ratio test

1) $\sum_{n=0}^{\infty} 5^{(-1)^n - n}$

it is a series with positive terms

root test:

$$\sqrt[n]{a_n} = \sqrt[n]{5^{((-1)^n - n)}} = 5^{\left(\frac{(-1)^n}{n} - 1\right)}$$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \frac{1}{5}$$

$\frac{1}{5} < 1 \implies$ the series converges

2) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^2(2n)!}$ ratio test

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(n+1)^2(2n+2)!} \cdot \frac{n^2(2n)!}{(n!)^2} =$$

$$= (n+1)^2 \left(\frac{n}{n+1}\right)^2 \frac{1}{(2n+2)(2n+1)}$$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n^2+2n+1}{4n^2+3n+2} = \frac{1}{4} < 1$$

By ratio test the series converges.

3) $\sum_{n=1}^{\infty} \frac{n^k}{n!}$ ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^k}{(n+1)!} \cdot \frac{n!}{n^k} = \left(1+\frac{1}{n}\right)^k \frac{1}{n+1} \xrightarrow{n\to+\infty} 0$$

the series converges

4) $$\sum_{n=0}^{\infty} \frac{n^{n+1}}{3^n (n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+2}}{3^{n+1}(n+2)!} \cdot \frac{3^n (n+1)!}{n^{n+1}} =$$

$$= \frac{n+1}{3}\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) \frac{1}{(n+2)} =$$

$$= \frac{n+1}{3(n+2)} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)^n \longrightarrow \frac{e}{3} < 1$$

$\Longrightarrow$ The series converges

5) For which value $x \in \mathbb{R}$ does $\sum_{n=1}^{\infty} \frac{n^{nx}}{n!}$ converge?

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{(n+1)x}}{(n+1)\cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{n^{nx}} =$$

$$= (n+1)^{x-1} \left(\frac{n+1}{n}\right)^{nx}$$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} (n+1)^{x-1} \left(\left(1+\frac{1}{n}\right)^n\right)^x =$$

$$e^x \cdot \lim_{n\to\infty} (n+1)^{x-1} = \begin{cases} +\infty & \text{if } x>1 \\ e & \text{if } x=1 \\ 0 & \text{if } x<1 \end{cases}$$

therefore the series converges if and only if $x<1$

(Since the series has positive terms, it diverges if and only if $x \geq 1$)

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(6) $\sum_{n=1}^{\infty} \frac{(n!)^x}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^x}{(n+1)^{n+1}} \frac{n^n}{(n!)^x} = \frac{(n+1)^x}{(n+1)\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{(n+1)^{x-1}}{e} = \begin{cases} +\infty & \text{if } x>1 \\ \frac{1}{e} & \text{if } x=1 \\ 0 & \text{if } x<1 \end{cases}$$

By the ratio test the series converges if and only if $x \leq 1$

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(7) $\sum_{n=1}^{\infty} x^{n!}$ Study convergence as $x \geq 0$

It is immediate to see that for $x \geq 1$ the sequence a_n is not infinitesimal (i.e. $\lim a_n \neq 0$), so for $x \geq 1$ it cannot converge.

root test:

$$\sqrt[n]{a_n} = x^{\frac{n!}{n}} = x^{(n-1)!}$$

Now $\lim_{x \to +\infty} x^{(n-1)!} = 0$ if $0 \leq x < 1$

So the series converges if and only if $x \in [0, 1[$.

$\sum_{n=1}^{\infty} n!\, x^n$ Study convergence as the parameter $x \geq 0$.

Let us try the root test:

$$\sqrt[n]{a_n} = x (n!)^{\frac{1}{n}} = x\, n^{\frac{1}{n}} \cdot (n-1)^{\frac{1}{n}} \cdots (2)^{\frac{1}{n}}$$

$$x (n-1)\, 2^{\frac{1}{n}} \xrightarrow[n \to +\infty]{} \begin{cases} +\infty & x > 0 \\ 0 & x = 0 \end{cases}$$

Let us also try the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)x \xrightarrow[n \to +\infty]{} \begin{cases} +\infty & x > 0 \\ 0 & x = 0 \end{cases}$$

Both tests say that the serie converges $\Leftrightarrow$ $x = 0$

Exercise 2.9.5

1)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1} = \sum_{n=1}^{\infty} (-1)^n a_n \qquad a_n = \frac{n}{n^2+1}$$

Alternating series. Try Leibniz:

- a_n is infinitesimal:
  $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$
- a_n is decreasing: $\exists \bar{n}$ s.t. $a_{n+1} \leq a_n \; \forall n \geq \bar{n}$?

$$\frac{n+1}{(n+1)^2+1} \leq \frac{n}{n^2+1}$$

$$\Updownarrow$$

$$n^3 + \cancel{n} + n^2 + 1 \leq (n+1)^2 n + \cancel{n}$$

$$\Updownarrow$$

$$\cancel{n^3} + n^2 + 1 \leq (n^2+1+2n)n = \cancel{n^3} + 2 + 2n^2$$

$$n^2 + 1 \geq 0$$

yes, for every $n \geq 0$ (so $\bar{n} = 0$)

$$\xRightarrow{\text{Leibniz}} \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1} \text{ converges.}$$

— However it does not converge absolutely:

$$\sum \left|(-1)^n a_n\right| = \sum |a_n| = \sum \frac{n}{n^2+1}$$

and $\frac{n}{n^2+1} \sim \frac{1}{n}$, i.e. $\sum \frac{n}{n^2+1}$ does not converge

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2) $\sum (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{n}$

- $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$ $\quad \lim a_n \overset{?}{=} 0$

$$\lim_{n\to+\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \lim_{n\to+\infty} \frac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})n} =$$

$$\lim_{n\to\infty} \frac{n+1-n}{(\sqrt{n+1}+\sqrt{n})n} = \lim_{n\to+\infty} \frac{1}{(\sqrt{n+1}+\sqrt{n})n} = 0$$

- a_n is decreasing?

$$a_{n+1} \overset{?}{\leq} a_n$$

$$\frac{\sqrt{n+2}-\sqrt{n+1}}{n+1} \leq \frac{\sqrt{n+1}-\sqrt{n}}{n}$$

$$\Updownarrow$$

$$\frac{1}{(\sqrt{n+2}+\sqrt{n+1})(n+1)} \leq \frac{1}{((\sqrt{n+1})+\sqrt{n})n}$$

$$\Updownarrow$$

$$(\sqrt{n+1}+\sqrt{n})n \leq (\sqrt{n+2}+\sqrt{n+1})(n+1)$$

which is clearly true $\forall n \in \mathbb{N}$

Hence, by Leibniz test, the series does converge.

Does it converge absolutely?

$$\sum_{n=1}^{\infty}\left|(-1)^n a_n\right| = \sum_{n=1}^{\infty} a_n = \sum \frac{\sqrt{n+1}-\sqrt{n}}{n} =$$

$$= \sum \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} = \sum \frac{1}{n^{\frac{3}{2}}\left(\sqrt{1+\frac{1}{n}}-1\right)} =$$

$$= \sum \frac{1}{n^{\frac{3}{2}}\left(\cancel{1}+\frac{1}{2n}+o\left(\frac{1}{2n}\right)-\cancel{1}\right)}$$

By $a_n = \frac{1}{n^{\frac{3}{2}}\left(\frac{1}{2n}+o\left(\frac{1}{2n}\right)\right)} \sim \frac{2}{n^{\frac{1}{2}}}$, since $\frac{1}{2}<1$

the series does not converge absolutely

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt[n]{3}-1\right) = \sum_{n=1}^{\infty}(-1)^n a_n$$

$a_n = \sqrt[n]{3}-1$

- $\lim a_n = \sqrt{3}-1=0$
- a_n decreasing:

$$a_{n+1}-a_n = \sqrt[n+1]{3}\,\cancel{-1}-\sqrt[n]{3}\,\cancel{+1}<0$$

$$\overset{\text{Leibniz}}{\Longrightarrow} \sum(-1)^n a_n \text{ converges.}$$

Does it converge absolutely?

Try ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{3^{\frac{1}{n+1}} - 1}{3^{\frac{1}{n}} - 1} = \frac{\cancel{3^{\frac{1}{n}}}\left(3^{\frac{1}{n+1}-\frac{1}{n}} - 3^{-\frac{1}{n}}\right)}{\cancel{3^{\frac{1}{n}}}\left(1 - 3^{-\frac{1}{n}}\right)} =$$

$= 0$

yes, it converges absolutely.

($\Rightarrow$ converges, but we already knew that by the previous point)

5)

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}-1}}{\cos(n\pi)}$$

we recognize an alternating series:

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}-1}}{\cos(n\pi)} = \sum_{n=1}^{\infty} (-1)^n e^{\frac{1}{n}-1}$$

Now

$$a_n = e^{\frac{1}{n}-1} \xrightarrow{n\to+\infty} \frac{1}{e}.$$

The generic term (both a_n and $(-1)^n a_n$) is not infinitesimal,

so the series does not converge.

$\Longrightarrow \sum a_n$ diverge.

As for $\sum (-1)^n a_n$, it is

indeterminate (Prove it by compute the even partial sums S_{2k} and the odd ones S_{2k+1})

9) $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{4^n}$

Let us study the absolute convergence, i.e. the convergence of $\sum_{n=0}^{\infty} \frac{n+1}{4^n}$

By $\frac{a_{n+1}}{a_n} = \frac{n+2}{4\,\cancel{4^{n}}} \cdot \frac{\cancel{4^n}}{n+1} \xrightarrow{n\to\infty} \frac{1}{4} < 1 \Rightarrow$

$\sum_{n=1}^{\infty} \frac{n+1}{4^n}$ converges, so our series converges absolutely, hence it converges.

11) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

Converge absolutely? i.e., does $\sum_{n=1}^{\infty} \frac{\log n}{n}$ converge? Try ratio test:

$$\frac{\log (n+1)}{n+1} \cdot \frac{n}{\log n} = \frac{n}{n+1} \cdot \frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n} = 1$$

So the ratio test does not tell us anything. Let us try the root test:

$$\sqrt[n]{\frac{\log n}{2n}} = e^{\frac{1}{n}\log\left(\frac{\log n}{2n}\right)} = e^{\frac{1}{n}\left(\log(\log n) - \log n\right)}$$

By $\lim \dfrac{\log\log x - \log x}{x} \underset{\text{Hôpital}}{=} \lim \dfrac{\frac{1}{\log x}\cdot\frac{1}{x} - \frac{1}{x}}{1} = 0$.

we deduce

$$\lim_{n\to\infty} \sqrt[n]{\frac{\log n}{2n}} = \lim_{n\to+\infty} e^{\frac{1}{n}\left(\frac{\log n}{2n}\right)} = 1.$$

So the root test as well tells us nothing about the absolute convergence...

But:

$\sum_{n=1}^{\infty} \frac{\log n}{2n}$ diverges by a simple comparison test:

indeed

$$\frac{\log n}{2n} > \frac{1}{n}, \text{ and } \sum \frac{1}{n}$$

diverges.

So our series $\sum (-1)^n \frac{\log n}{2n}$ does not converge absolutely.

It might converge ......

let us try Leibniz

- $a_n = \frac{\log n}{n} \to 0$ So a_n is infinitesimal.

- $a_{n+1} \leq a_n$?

Let us study the derivative

$$\left(\frac{\log x}{x}\right)' = \frac{1 - \log x}{x^2} \leq 0$$

$$\Longleftrightarrow \quad 1 \leq \log x \Longleftrightarrow x \geq e$$

So, yes, (a_n) is decreasing (for $n > 3$)

$$\underset{\text{Leibniz}}{\Longrightarrow} \quad \sum (-1)^n \frac{\log n}{n}$$

converges.