Padova, 2022 December 21

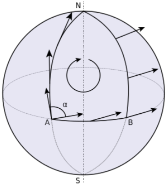
Master's degree course in

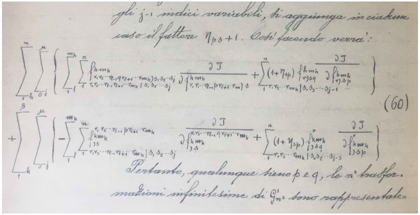
CONTROL SYSTEMS ENGINEERING

Olga Bernardi: **MATHEMATICAL PHYSICS**

Tullio Levi-Civita and the Parallel Transport Theory

Franco Cardin




gli j-1 indici variabili, si aggiunga in ciascun caso il fattore $\eta_{ps}+1$. Così facendo verrà:

(60)

Pertanto, qualunque sieno p e q, le n² trasfor- mazioni infinitesime di G_{n^2} sono rappresentate

Il 21 Novembre 2016 il Dipartimento di Matematica è stato intitolato al matematico Tullio Levi-Civita (Padova, 29 marzo 1873 – Roma, 29 dicembre 1941). Tullio Levi-Civita, che fu studente e (1897-1919) professore presso il nostro Ateneo, è un nome di cui i matematici padovani sono fieri, come lo sono per quello di Galileo Galilei. Nella primavera del 1915, a pochi mesi dalla pubblicazione della Relatività Generale, Levi-Civita si accorse di un errore che ne avrebbe minato la validità e scrisse ad Einstein per segnalarglielo. Einstein, dopo una iniziale riluttanza, riconobbe la correttezza delle obiezioni del padovano, a cui rimase grato per tutta la vita ("spaghetti and Levi-Civita" fu la risposta data da Einstein a un cronista che gli chiedeva cosa amasse di più dell'Italia). Levi-Civita si occupò di molti altri argomenti: la teoria del Trasporto Parallelo, che porta il suo nome, risulterà l'elemento fondante della moderna teoria delle Connessioni; oltre alla geometria della Relatività Generale, si occupò dalla stabilità delle orbite dei pianeti fino alla matematica dei cavi intercontinentali del telegrafo.

Le leggi di "difesa della razza" del 1938 lo allontanarono definitivamente dall'università e da tutte le istituzioni scientifiche italiane, mentre nel mondo libero se ne celebrava la grandezza.

TULLIO LEVI-CIVITA

He was born on 29 March **1873** in Padova, via Daniele Manin, 7.
Later, until the end of 1918, he lived in via Altinate, 14.

via Daniele Manin, 7

via Altinate, 14

His father should be remembered: **Giacomo Levi-Civita**, 1846-1922. 'Garibaldino' in Aspromonte at 16, with Garibaldi again in 1866 in Trentino.

Sergente dei Cacciatori delle Alpi

Memorable fact: Giacomo is a young lawyer, in 1880 he definitively claimed for the city of Padova the **Giotto frescoes of the Scrovegni Chapel** (at that time property of Gradenigo family), which were about to be detached and sold to a british museum.

He will be a beloved Mayor of Padova from 1904 to 1910, Senator of the Kingdom of Italy from 1908.

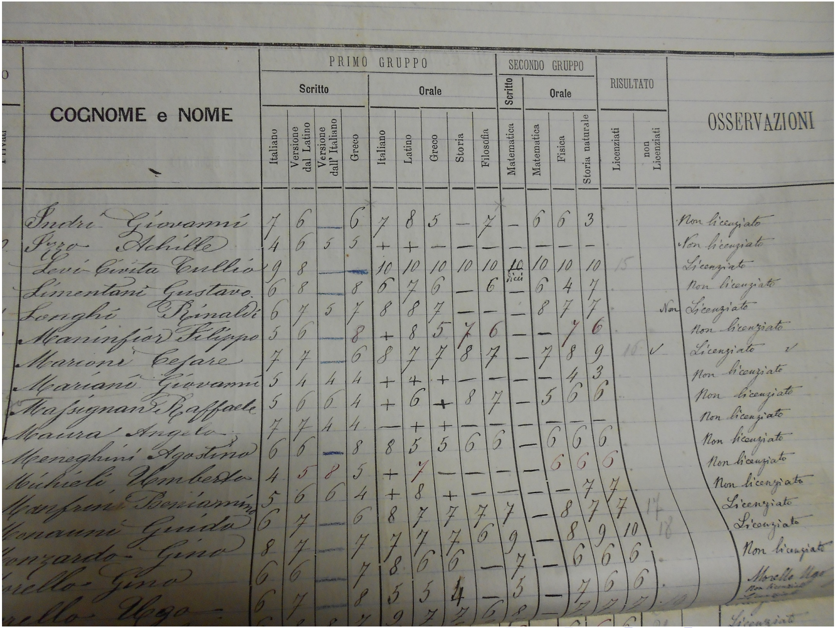
Tullio Levi-Civita attends the **Liceo Tito Livio**.
As professor of Mathematics: **Paolo Gazzaniga**, a very effective teacher and skilled on Number Theory.

The young Tullio feels himself "ready" to demonstrate the independence, or otherwise, of Euclid's 5^o Postulate from the other 4!

Gazzaniga does not discourage him, he follows him as a tutor and teacher, aware that the important 'disappointment' will become a growth for his pupil.

Gazzaniga will still be his professor at the University!

High school graduation, July 1890: Tullio was one of the very few students who opted for the writing of Mathematics!



COGNOME e NOME	PRIMO GRUPPO – Scritto: Italiano	Versione dal Latino	Versione dall'Italiano	Greco	PRIMO GRUPPO – Orale: Italiano	Latino	Greco	Storia	Filosofia	SECONDO GRUPPO – Scritto: Matematica	SECONDO GRUPPO – Orale: Matematica	Fisica	Storia naturale	RISULTATO: Licenziati	non Licenziati	OSSERVAZIONI
Andri Giovanni	7	6	—	6	7	8	5	—	7	—	6	6	3			Non licenziato
Pozzo Achille	4	6	5	5	+	+	—	—	—	—	—	—	—			Non licenziato
Levi-Civita Tullio	9	8	—	—	10	10	10	10	10	10 Dieci	10	10	10	15		Licenziato
Limentani Gustavo	6	8	—	8	6	7	6	—	6	—	6	4	7			Non licenziato
Longhi Rinaldi	6	7	5	7	8	8	7	—	—	—	8	7	7			Non Licenziato
Maninfior Filippo	5	6	—	8	+	8	5	7	6	—	—	7	6			Non licenziato
Maroni Cesare	7	7	—	6	8	7	7	8	7	—	7	8	9	16	✓	Licenziato
Mariani Giovanni	5	4	4	4	+	+	+	—	—	—	—	4	3			Non licenziato
Massignan Raffaele	5	6	6	4	+	6	+	8	7	—	5	6	6			Non licenziato
Maura Angelo	7	7	4	4	—	+	+	—	—	—	—	—	—			Non licenziato
Meneghini Agostino	6	6	—	8	8	5	5	6	6	—	6	6	6			Non licenziato
Michieli Umberto	4	5	8	5	+	7	—	—	—	—	6	6	6			Non licenziato
Manfrini Beniamino	5	6	6	4	+	8	+	—	—	—	—	7	7			Non licenziato

Tullio Levi-Civita studies Mathematics in Padova

Here, the great meeting of his scientific life, under the direction of

Gregorio Ricci Curbastro

he graduated in Mathematics in **1894**
Title of his thesis: *"Sugli Invarianti Assoluti"*

Sugli invarianti assoluti

1. Sia un sistema S di funzioni $f_1, f_2, \ldots, f_N$ dipendenti da n variabili $x_1, x_2, \ldots, x_n$, e si sappia che le leggi caratteristiche, le quali reggono le trasformazioni del sistema, sono l'invarianza, la covarianza, la contravarianza. Si tratta di:
« Determinare tutte le espressioni J, formate »
« colle variabili, colle funzioni del sistema, col- »
« le loro derivate ed integrali, le quali non cam- »
« biano di valore, quando sulle variabili e cor- »
« rispondentemente sulle funzioni, si eseguisce »
« una trasformazione arbitraria. »

Queste espressioni J si diranno invarianti assoluti del sistema S. Se precisamente d'ordine μ, se è μ l'ordine più elevato delle derivate in esse contenute; possono fin d'ora essere distinti in differenziali ed integrali, secondochè contengono esclusivamente le derivate delle funzioni del sistema od anche loro integrali.

2. Gauss fu il primo, che, riferendosi ad un particolare sistema, ne determinò un invariante differenziale e ne fece rilevare tutta l'im-

e ciò perchè $\varepsilon_{r_\ell p} = 0$, $\varepsilon_{r_\ell q} = 0$ per r_ℓ diversa da p o da q. Nel secondo e quarto termine invece, una delle s, s_j per es., può essere senz'altro fissata eguale a p, perchè, se nessuna s possiede tale valore, $\varepsilon_{p s_\lambda}$ è nullo per ogni valore di λ; corrispondentemente poi ad ogni combinazione con ripetizione delle $(j-1)$ s rimaste variabili, le somme

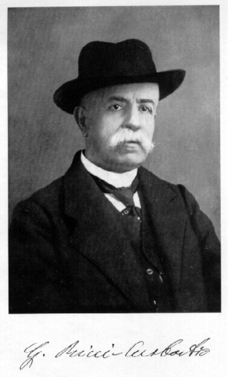
$$\sum_{1}^{j}{}_{\lambda} \varepsilon_{p s_\lambda} f^{h m_h}_{r_1 r_2 \cdots r_{m_h}) s_1 s_2 \cdots s_{\lambda-1} q s_{\lambda+1} \cdots s_j} \quad \text{ovvero} \quad \sum_{1}^{j} \varepsilon_{p s_\lambda} f^{r_1 r_2 \cdots r_{m_k}}_{k m_k) s_1 s_2 \cdots s_{\lambda-1} q s_{\lambda+1} \cdots s_j}$$

tenendo presente la simmetria degli indici s, saranno eguali a $f^{h m_h}_{r_1 r_2 \cdots r_{m_h}) s_1 s_2 \cdots s_{j-1} q}$ ovvero a $f^{r_1 r_2 \cdots r_{m_k}}_{k m_k) s_1 s_2 \cdots s_{j-1} q}$ più tante volte termini identici a quelli, quanti tra gli $j-1$ indici rimasti variabili sono eguali a p. Potremo adunque togliere il simbolo sommatorio rispetto a λ, purchè, fissato uno degli indici eguale a q, e continuando a chiamare s gli $j-1$ indici variabili, si aggiunga in ciascun caso il fattore $\eta_{ps}+1$. Così facendo verrà:

$$\sum_{1}^{\alpha}{}_{h} \sum_{0}^{\mu}{}_{j} \left\{ \sum_{1}^{m_h}{}_{\ell} \sum_{1}^{n}{}_{r_1 r_2 \cdots r_{\ell-1} r_{\ell+1} \cdots r_{m_h} | s_1 s_2 \cdots s_j} f^{h m_h}_{r_1 r_2 \cdots r_{\ell-1} q r_{\ell+1} \cdots r_{m_h}) s_1 s_2 \cdots s_j} \frac{\partial J}{\partial f^{h m_h}_{r_1 r_2 \cdots r_{\ell-1} p r_{\ell+1} \cdots r_m) s}} + \sum_{1}^{n}{}_{r_1 r_2 \cdots r_{m_h} | s_1 s_2 \cdots s_{j-1}} (1+\eta_{sp}) f^{h m_h}_{r) s q} \frac{\partial J}{\partial f^{h m_h}_{r) s p}} \right\}$$

$$+ \sum_{1}^{\beta}{}_{k} \sum_{0}^{\mu}{}_{j} \left\{ - \sum_{1}^{m_k}{}_{\ell} \sum_{1}^{n}{}_{r_1 r_2 \cdots r_{\ell-1} r_{\ell+1} \cdots r_{m_k} | s_1 s_2 \cdots s_j} f^{r_1 r_2 \cdots r_{\ell-1} p r_{\ell+1} \cdots r_{m_k}}_{k m_k) s} \frac{\partial J}{\partial f^{r_1 r_2 \cdots r_{\ell-1} q r_{\ell+1} \cdots r_{m_k}}_{k m_k) s}} + \sum_{1}^{n}{}_{r_1 r_2 \cdots r_{m_k} | s_1 s_2 \cdots s_{j-1}} (1+\eta_{sp}) f^{r}_{k m_k) s q} \frac{\partial J}{\partial f^{r}_{k m_k) s p}} \right\} \qquad (60)$$

Pertanto, qualunque sieno p e q, le n^2 trasformazioni infinitesime di G'_{n^2} sono rappresentate



Gregorio Ricci Curbastro
(Lugo di Romagna, 12 Jan 1853 - Bologna, 6 Aug 1925):
Student at the S. Normale di Pisa of Enrico Betti, Eugenio Beltrami, Ulisse Dini.
As soon as he graduated, he goes on a study trip to **Munich** (a kind of Post-doc!):
There, he meets Felix Klein, learns the new recent geometric theory on

CURVATURE

introduced by **Bernhard Riemann**, who was the great pupil of **Carl Friedrich Gauss.**

Here a list of the papers by Levi-Civita in the **first two years, 1894-96,** after his Math Graduation:

Tullio Levi-Civita, Sugli infiniti ed infinitesimi attuali quali elementi analitici, Atti del R. Istituto Veneto, 1893.
Tullio Levi-Civita, Sugli invarianti assoluti [Dissertazione di laurea] Atti del R. Istituto Veneto, 1893-94.
Tullio Levi-Civita, Sui gruppi di operazioni funzionali, Rendiconti del R. Istituto lombardo di scienze e lettere,1895.
Tullio Levi-Civita, Alcune osservazioni alla nota Sui gruppi di operazioni funzionali, Rendiconti del R. Istituto lombardo di scienze e lettere,1895.
Tullio Levi-Civita, I gruppi di operazioni funzionali e l'inversione degli integrali definiti (Rendiconti del R. Istituto lombardo di scienze e lettere, 1895.
Tullio Levi-Civita, Di una espressione analitica atta a rappresentare il numero dei numeri primi compresi in un determinato intervallo (Atti della R. Accademia dei Lincei. Rendiconti della Classe di scienze fisiche, matematiche e naturali, 1895.
Tullio Levi-Civita, Sull'inversione degli integrali definiti nel campo reale (Atti della R. Accademia delle scienze di Torino, 1895-96.
Tullio Levi-Civita, Sulla distribuzione indotta in un cilindro indefinito da un sistema simmetrico di masse, Atti della R. Accademia dei Lincei. Rendiconti della Classe di scienze fisiche, matematiche e naturali, 1895.
Tullio Levi-Civita, Sugli integrali algebrici delle equazioni dinamiche (Atti della R. Accademia delle scienze di Torino, 1895-96.
Tullio Levi-Civita, Sulle trasformazioni delle equazioni dinamiche (Annali di Mat. pura ed applicata, 1896.

In 1898, at age of 25, Levi-Civita became the holder of the chair of Rational Mechanics in Padova.

The team Ricci & Levi-Civita is wonderful!
there is a flourishing of modern Differential Geometry, Tensor Calculus, Theory of Invariants ... they called all this the

Absolute Differential Calculus

Covariante Derivative: it was introduced by Gregorio Ricci Curbastro in 1888, later developed with Tullio Levi-Civita in Riemannian metric theory, up to the definitive arrangement in the important review article, a booklet of 76 pages, requested by Felix Klein:

G. Ricci Curbastro, T. Levi-Civita: Methodes de calcul differentiel absolu et leurs applications, Mathematische Annalen 54 (1-2): 125-201, 1901.

Ricci and Levi-Civita (following the ideas of Elwin Bruno Christoffel) observed in fact that Christoffel's symbols, which they used to define curvature, could have provided a notion of differentiation that generalized
the classical derivative of a **vector field Y** along the direction of **another vector field X,**

- Classically: $\frac{d}{dt}Y^\alpha(x(t)) = \frac{\partial Y^\alpha}{\partial x^\beta}\frac{dx^\beta}{dt} = \underbrace{\frac{\partial Y^\alpha}{\partial x^\beta}X^\beta}$,

$\frac{dx^\beta(t)}{dt} = X^\beta(x(t))$

- Covariant Derivative:

$$(D_X Y)^\alpha = \left(D_{\frac{\partial}{\partial x^\beta}} Y^\alpha\right) X^\beta := \sum_\beta \left(\underbrace{\frac{\partial Y^\alpha}{\partial x^\beta}}_{\text{standard term}} + \underbrace{\Gamma^{\ \alpha}_{\beta\gamma}\, Y^\gamma}_{\text{new term}} \right) X^\beta$$

$$\underbrace{\Gamma^{\ \alpha}_{\beta\gamma}}_{\text{Christoffel's symbols of } 2^0 \text{ kind}} = \sum_\rho \underbrace{\frac{1}{2}\left(g_{\rho\beta,\gamma} + g_{\gamma\rho,\beta} - g_{\beta\gamma,\rho}\right)}_{[\beta\gamma,\rho]:\ \text{Christoffel's symbols of } 1^0 \text{ kind}} g^{\rho\alpha}$$

This new derivative turned out to be **covariant**, in the sense that it satisfied Riemann's requirement that objects in geometry must be independent of their description in a particular coordinate system.

But there was a problem.

That wonderful theory was received (very) coldly by the math community :
Tullio Levi-Civita, from that year 1901, stopped researching Differential Geometry,
and substantially, so did his master Gregorio Ricci Curbastro.

Tullio L-C came back to dealing with Differential Geometry precisely with his mail correspondence with Einstein, in 1915, arriving after at the general definition of Parallel Transport in 1916. (Sketched later)

In those first 15 years of the new century Tullio studied and published admirable researches in

- theory of stability,
- analytical mechanics

and mainly in

- celestial mechanics, the Newtonian **Three-Body Problem**.

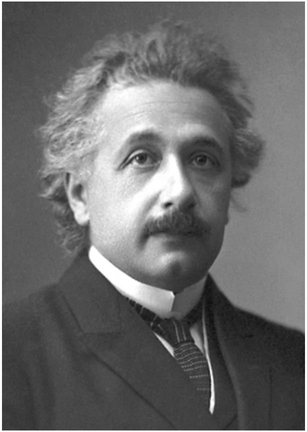
He anticipates, in 1899, although restricted to the finite-dimensional case only, the well-known **Theorem** by **Emmy Noether** of 1918:

SYMMETRIES $\Longrightarrow$ FIRST INTEGRALS

INTERPRETAZIONE GRUPPALE DEGLI INTEGRALI DI UN SISTEMA CANONICO (*)

« Rend. Acc. Lincei », s. 3ª, vol. VIII, 2º sem. 1899, pp. 235-238

But 1915 is coming:
Albert Einstein, Berlin



General Relativity , *before* Levi-Civita:

Einstein A., 1912, Zur Theorie des statischen Gravitationsfeldes Annalen der Physik, 38 (1912), pp. 443–458 (Reprinted as Vol. 4, Doc. 4 CPAE)

Einstein A., 1913, Zum gegenwrtigen Stande des Gravitationsproblems Physikalische Zeitschrift, 14 (1913), pp. 124–1266 (Reprinted as Vol. 4, Doc. 17 CPAE)

Einstein, A., & **Grossmann, M.** (1913). **Entwurf** einer verallgemeinerten Relativittstheorie und einer Theorie der Gravitation. Leipzig: Teubner (Reprinted in CPAE, Vol. 4, Doc. 13).

Einstein A., 1914, Die formale Grundlage der allgemeinen Relativittstheorie. Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften (pp. 79–801) (Reprinted as Vol. 6, Document 9 CPAE).

Entwurf : draft, sketch

Another character:
Max Abraham, Politecnico di Milano

Max Abraham, Polytechnic University of Milano, is a 'classical physicist',
he had been a strong supporter of the existence of the *ether* and that the electron was a perfectly rigid sphere with a charge uniformly distributed on its surface.

At the beginning of **1915**, in a meeting with Tullio Levi-Civita, old friend of him, Max invites Tullio to read and then to 'criticize' the **Entwurf**.

But for Tullio Levi-Civita is instead a shock:
He is now reading the theory of his teacher Ricci Curbastro and himself,
that theory 'becomes physics'!
Tullio immediately writes to Einstein **(we are in March)**:

"It's a beautiful theory, but there is a mistake!"

Einstein denies, the correspondence is very dense.

At the end, Einstein admits **(it's May 5)**, and thanks Tullio,
but the correspondence broke off dramatically:
On **May 7** Italy left the 'Triple Alliance'
and on **May 24** it declared war on Austria.

But now, the gravitational equations are finally correct !

Relatività Generale, *After* Levi-Civita:

Einstein A., 1916a, Die Grundlage der allgemeinen Relativittstheorie Annalen der Physik, 49 (7) (1916), pp. 769–822 (Reprinted as Vol. 6, Doc. 30 CPAE)

Einstein A., 1916b, Uber Friedrich Kottlers Abhandlung ber Einsteins quivalenzhypothese und die Gravitation Annalen der Physik, 356 (22) (1916), pp. 639–642 (Reprinted as Vol. 6, Doc. 40 CPAE)

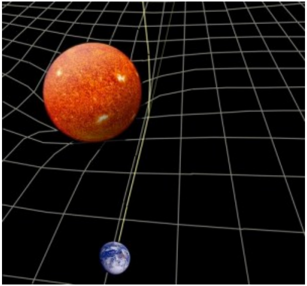
Einstein warmly expresses his gratitude to Levi Civita !

Einstein, to the question "what does he love about Italy?" will always answer: "spaghetti and Levi-Civita".

Equations of General Relativity:

$$\underbrace{A^{\alpha\beta}}_{\text{“Levi-Civita” curvature tensor}} = \frac{8\pi\gamma}{c^4} \underbrace{\mathcal{U}^{\alpha\beta}}_{\text{matter}}$$

$$\left(A^{\alpha\beta} = A^{\alpha\beta}(g_{\rho\sigma}\ ,\ g_{\rho\sigma,\gamma}\ ,\ g_{\rho\sigma,\gamma\delta})\right)$$



December 1916: Tullio Levi-Civita introduces the

Parallel Transport Theory

Francesco Severi,
a master of italian algebraic geometry

There was a friendship between Francesco Severi and Tullio Levi Civita.
But Severi changed sides, became a fascist, attended scientific conferences in a black shirt and orbace.
Severi suggested to Minister Gentile to oblige university professors to take an oath of allegiance to the regime, 1931

Fig. 1. – Francesco Severi accompagna Benito Mussolini nella visita alla Biblioteca dell'Istituto di Matematica dell'Università di Roma; nello sfondo, a destra, si riconosce Enrico Bompiani.

The **fascist racial laws** against Jews arrived in 1938.

By explicit intervention of Francesco Severi,

it was forbidden to Castelnuovo, Enriques e Levi-Civita, **the day after the promulgation of the racial laws**, to enter the library of the Mathematical Institute of Rome.

from: Giorgio Israel e Pietro Nastasi, Scienza e razza nell'Italia fascista (Bologna, Il Mulino, 1998), see p. 258

In the next slide, the letter of dismissal of Levi-Civita.

Mod. 3

Regia Università degli Studi di Roma

Roma, addì 25 OTT 1938 Anno XVI 19 Anno

Chiar.mo Signor
Prof. Tullio Levi-Civita
ROMA

Pos. N.° 4 Prot. N.° 9977
Allegati
Risposta al foglio del
Pos. N.° Prot. N.°

OGGETTO : Personale di razza ebraica.

Dalla Vostra scheda di censimento personale risulta che appartenete alla razza ebraica.

Siete stato, pertanto, sospeso dal servizio a decorrere dal I6 ottobre I938 XVI a norma del R.D.L. 5-9-I938 nº 1390.

Con osservanza,

p. IL RETTORE
G. Cardinali

(…)isposta si prega di citare la data e i numeri della presente

Levi-Civita dies in 1941 December 29 in Roma.

November 21, 2016:

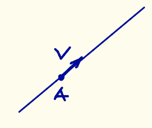
The Department of Mathematics was intitolated to Tullio Levi-Civita

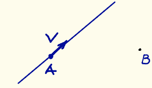
Now, some ideas about the

Parallel Transport Theory

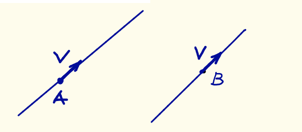
In an Euclidean 2-space $\mathbb{E}$, a plane, take a point A and an *applied vector* $\mathbf{V}$ attached to A.
Naturally, we describe the straight line:

$$A + \mathbb{R}\,\mathbf{V}$$



Now, take another point B: 

, by the 5^o Postulate we can consider the **unique** straight line through B which is **parallel** to the above straight line $A + \mathbb{R}\,\mathbf{V}$

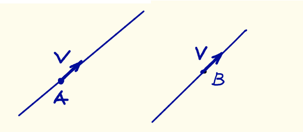


We speak of **equivalent** (**equipollenti**) vectors and we can write this parallel straight line through B, simply by: $B + \mathbb{R}\,\mathbf{V}$

Look: we are really using the **same** symbol $\mathbf{V}$ for two **equivalent** vectors, but rigorously they are **different**, because attached to different points!

We will say: the vector **$\mathbf{V}$, attached to B,** is the **parallelly transported** of the vector **$\mathbf{V}$, attached to A**:

$$(A, \mathbf{V}) \in \underbrace{T_A\mathbb{E}}_{\textit{tangent space at } A} \quad \rightsquigarrow \quad (B, \mathbf{V}) \in \underbrace{T_B\mathbb{E}}_{\textit{tangent space at } B}$$



The above discussion could seem a little trivial,

but all becomes cumbersome and intriguing

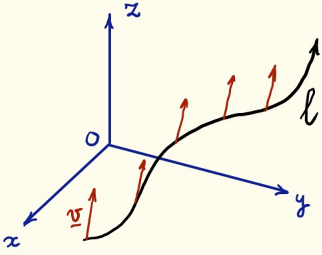
when we try to extend these arguments

from the Euclidean 2-plane, or Euclidean 3-space, to a generic,

fully non linear, 2-surface Σ!

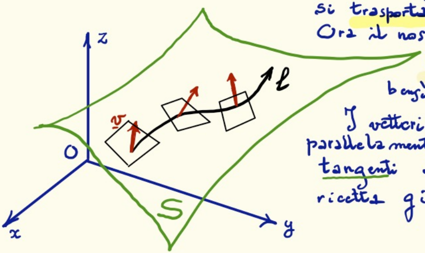
Trasporto Parallelo

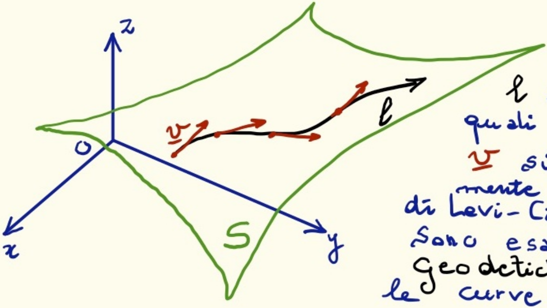
di un vettore $\mathbf{v}$ lungo una curva ℓ in uno spazio vettoriale Euclideo (O, x, y, z)



Se ora la curva ℓ è sopra ad una superficie S, come si trasporta il vettore $\mathbf{v}$? Ora il nostro "universo" non è $\mathbb{R}^3 = \{O, x, y, z\}$, bensì S.

I vettori trasportati parallelamente dovranno essere tangenti a S. Ma qual è la ricetta giusta? (Levi-Civita)





Altro problema:
Quali sono le curve ℓ sopra S per le quali il vettore tangente $\underline{v}$ si trasporta parallelamente, secondo la ricetta di Levi-Civita?

Sono esattamente le geodetiche di S, cioè le curve di lunghezza minima!

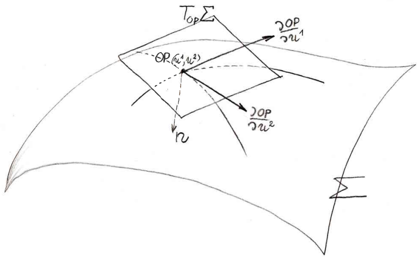
Let consider a 2-surface $\Sigma \subset \mathbb{R}^3$ by parametric equation:

$$\mathbb{R}^2 \supset U \ni (u^1, u^2) \longmapsto OP(u^1, u^2) \in \mathbb{R}^3, \quad rank\, dOP\big|_U = max(= 2)\,,$$

the triple of vectors:

$$\left(\frac{\partial OP}{\partial u^1}, \frac{\partial OP}{\partial u^2}, n := \frac{\frac{\partial OP}{\partial u^1} \times \frac{\partial OP}{\partial u^2}}{\left\| \frac{\partial OP}{\partial u^1} \times \frac{\partial OP}{\partial u^2} \right\|} \right)$$

is linearly independent in $\mathbb{R}^3$,
the pair $\left(\frac{\partial OP}{\partial u^1}, \frac{\partial OP}{\partial u^2} \right)$ is a basis of the tangent space $T_{OP}\Sigma$



Main Problem:

Given

(i) a curve over Σ, $\ell : [0,1] \ni \lambda \to \ell(\lambda) = u^\alpha(\lambda)|_{\alpha=1,2} \in \Sigma$,

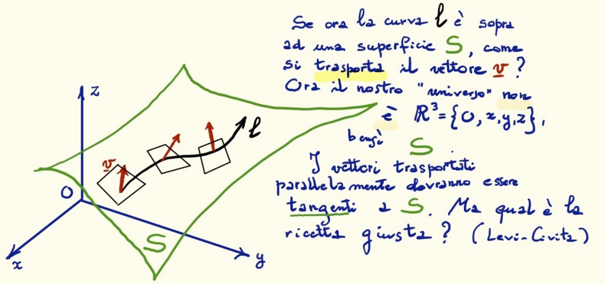
and

(ii) a vector $V_0 \in T_{\ell(0)}\Sigma$,

what is the 'reasonable' curve of vectors

$$\tilde{\ell} : [0,1] \ni \lambda \to \tilde{\ell}(\lambda) = (\ell(\lambda), V(\lambda)) \in T\Sigma \qquad V(0) = V_0$$

parallelly transporting V_0 along ℓ?



A first 'naive' trial (rem: we want $V \cdot n \equiv 0$, that is, $V \in T\Sigma$)

$$V(\lambda): \qquad V(\lambda) = V_0 - \Big(V_0 \cdot n(\lambda)\Big) n(\lambda), \tag{0.1}$$

$\Big(n(\lambda)$ means $n(\ell(\lambda))\Big)$ in other words, we are transporting V_0 along the curve ℓ by using the 'Euclidean' structure of $\mathbb{R}^3$, removing, point after point, the normal component of V_0.
The derivative with respect to λ is

$$\dot{V} = -(V_0 \cdot \dot{n})n - (V_0 \cdot n)\dot{n}$$

Since $n \cdot \dot{n} \equiv 0$, by multiplying (0.1) by $\dot{n}$, we have $V \cdot \dot{n} = V_0 \cdot \dot{n}$, so that

$$\dot{V} = -(V \cdot \dot{n})n - (V_0 \cdot n)\dot{n} \tag{0.2}$$

$$\dot{V} = -(V \cdot \dot{n})n - (V_0 \cdot n)\dot{n}$$

Drawbacks:

i) we are using heavily the Euclidean environment $\mathbb{R}^3$ in which Σ is embedded,

ii) the term $-(V_0 \cdot n)\dot{n}$ is definitely *non local*.

The solution proposal by Levi-Civita will be exactly that of neglecting
that last term $-(V_0 \cdot n)\dot{n}$, that is, just considering

$$\dot{V} = -(V \cdot \dot{n})n \qquad or: \quad 0 = \dot{V} \cdot \frac{\partial OP}{\partial u^\beta} \tag{0.3}$$

Multiplying by n each member:

$$\dot{V} \cdot n + V \cdot \dot{n} = \frac{d}{d\lambda}(V \cdot n) = 0$$

we restore again the main expected features:

$$V \cdot n \equiv 0 \qquad \text{since} \;\; V(0) \cdot n(0) = 0$$

Now we restart, by arriving to give an intrinsic formulation of the parallel transport, just using geometric objects related to Σ, forgetting $\mathbb{R}^3$, involving purely the Riemannian metric on Σ, even though it is inherited from the Euclidean metric by pull-back:

$$g_{\alpha\beta} := \frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial OP}{\partial u^\beta}, \tag{0.4}$$

$$V = v^\alpha \frac{\partial OP}{\partial u^\alpha}, \qquad V \cdot \frac{\partial OP}{\partial u^\beta} = v^\alpha \frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial OP}{\partial u^\beta} = v^\alpha g_{\alpha\beta} = v_\beta$$

On the basis of (0.3), i.e.:

$$\dot{V} = -(V \cdot \dot{n})n$$

avoiding any links to the environment $\mathbb{R}^3$ of Σ, following Levi Civita[1]:

$$V(\lambda): \qquad 0 = \dot{V} \cdot \frac{\partial OP}{\partial u^\beta}, \qquad \beta = 1, 2. \tag{0.5}$$

[1] Tullio Levi-Civita: Nozione di parallelismo in una varietà qualunque, Rend. Circ. Mat. Palermo 42 (1917), 173-205.

$$V(\lambda): \qquad 0 = \dot{V} \cdot \frac{\partial OP}{\partial u^\beta}, \qquad \beta = 1, 2.$$

Rewriting the latter one 'by parts',

$$0 = \frac{d}{d\lambda}\left(V \cdot \frac{\partial OP}{\partial u^\beta}\right) - V \cdot \frac{d}{d\lambda}\frac{\partial OP}{\partial u^\beta},$$

recalling the representation $V = v^\alpha \frac{\partial OP}{\partial u^\alpha}$ and $v_\beta = V \cdot \frac{\partial OP}{\partial u^\beta}$,

$$0 = \frac{dv_\beta}{d\lambda} - v^\alpha \underbrace{\frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial^2 OP}{\partial u^\gamma \partial u^\beta}}_{??} \frac{d\ell^\gamma}{d\lambda}$$

This last term $\frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial^2 OP}{\partial u^\gamma \partial u^\beta}$ is fully intrinsic, it forgets the embedding of Σ in $\mathbb{R}^3$!

Indeed, from (0.4) $g_{\alpha\beta} = \frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial OP}{\partial u^\beta}$, and writing: $g_{\alpha\beta,\gamma} = \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}$,

$$g_{\alpha\beta,\gamma} = \frac{\partial^2 OP}{\partial u^\gamma \partial u^\alpha} \cdot \frac{\partial OP}{\partial u^\beta} + \frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial^2 OP}{\partial u^\gamma \partial u^\beta}$$

$$g_{\beta\gamma,\alpha} = \frac{\partial^2 OP}{\partial u^\alpha \partial u^\beta} \cdot \frac{\partial OP}{\partial u^\gamma} + \frac{\partial OP}{\partial u^\beta} \cdot \frac{\partial^2 OP}{\partial u^\alpha \partial u^\gamma}$$

$$g_{\gamma\alpha,\beta} = \frac{\partial^2 OP}{\partial u^\beta \partial u^\gamma} \cdot \frac{\partial OP}{\partial u^\alpha} + \frac{\partial OP}{\partial u^\gamma} \cdot \frac{\partial^2 OP}{\partial u^\beta \partial u^\alpha}$$

finally,

$$\underbrace{\frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial^2 OP}{\partial u^\gamma \partial u^\beta}}_{!!} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} - g_{\beta\gamma,\alpha}\right) =: [\gamma\beta, \alpha]$$

$$\frac{\partial OP}{\partial u^\alpha} \cdot \frac{\partial^2 OP}{\partial u^\gamma \partial u^\beta} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} - g_{\beta\gamma,\alpha}\right) =: [\gamma\beta, \alpha]$$

The three-index objects $[\gamma\beta, \alpha]$ are precisely the

symbols of Christoffel of first kind!

We have seen, at the end, the condition $0 = \dot{V} \cdot \frac{\partial OP}{\partial u^\beta}$ becomes

$$0 = \frac{dv_\beta}{d\lambda} - v^\alpha [\gamma\beta, \alpha] \frac{d\ell^\gamma}{d\lambda} \qquad \text{or :} \quad \frac{Dv_\beta}{D\lambda} = 0 \tag{0.6}$$

which is exactly the vanishing condition for the

Covariant Derivative[2]

of the searched vector v along the assigned curve ℓ.

[2] A notion in advance introduced by Gregorio Ricci Curbastro e Tullio Levi-Civita at the end of the XIX century.

Self-parallel curves

Our last problem now is a little different:

we would like to characterize the curves $\lambda \mapsto \ell^\alpha(\lambda)$ on Σ such that their tangent vectors are parallel transported second the above recipe.

$$\underbrace{0 = \frac{dv_\beta}{d\lambda} - v^\alpha[\gamma\beta, \alpha]\,\frac{d\ell^\gamma}{d\lambda}}_{*}, \qquad \text{where :} \quad v^\beta = \frac{d\ell^\beta}{d\lambda} \tag{0.7}$$

by introducing the *symbols of Christoffel of second kind*:

$$\Gamma^{\ \sigma}_{\gamma\beta} := \frac{1}{2} g^{\sigma\alpha}\left(g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} - g_{\beta\gamma,\alpha}\right) = g^{\sigma\alpha}[\gamma\beta, \alpha]$$

$$\underbrace{\frac{d^2\ell^\sigma}{d\lambda^2} + \Gamma^{\ \sigma}_{\beta\gamma}(\ell)\frac{d\ell^\beta}{d\lambda}\frac{d\ell^\gamma}{d\lambda} = 0}_{**} \qquad \text{(geodesic equation)} \tag{0.8}$$

It is a nice exercise to discover the above equation are the **Euler-Lagrange** equation related to a **Variational Problem**: Look for the stationary curves of the **energy functional** between two fixed points x_0 and x_1 on a smooth manifold and without active forces:

$$\mathcal{E} : \mathcal{V}_{0,T}^{x_0,x_1} \longrightarrow \mathbb{R} \tag{0.9}$$

$$x(\cdot) \longmapsto \int_0^T \underbrace{\frac{1}{2}\, g_{\alpha\beta}(x(\lambda))\frac{dx^\alpha}{d\lambda}(\lambda)\frac{dx^\beta}{d\lambda}(\lambda)}_{L(x,\dot{x}) = \frac{1}{2}\, g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta = \text{Kinetic Energy, for } m=1} d\lambda$$

A popular fact:

if a curve $x(t)$ stationarizes the energy functional,
then it stationarizes the length functional:

$$\mathcal{L} : \mathcal{V}_{0,T}^{x_0,x_1} \longrightarrow \mathbb{R} \tag{0.10}$$

$$x(\cdot) \longmapsto \int_0^T \underbrace{\sqrt{g_{\alpha\beta}(x(\lambda))\frac{dx^\alpha}{d\lambda}(\lambda)\frac{dx^\beta}{d\lambda}(\lambda)}}_{\text{Euclidean Norm of the `Velocity'}} d\lambda$$

Compute $\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0$, **for** $L = \frac{1}{2}\, g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta$,

$$\frac{d}{d\lambda}\left(g_{\alpha\beta}(x)\frac{dx^\beta}{d\lambda}\right) - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}(x)\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0,$$

$$\underbrace{\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}}_{\star} + g_{\alpha\beta}\frac{d^2x^\beta}{d\lambda^2} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0,$$

$$g_{\alpha\beta}\frac{d^2x^\beta}{d\lambda^2} + \underbrace{\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} + \frac{1}{2}\frac{\partial g_{\gamma\alpha}}{\partial x^\beta}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}}_{\star} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0,$$

$$\frac{d^2x^\sigma}{d\lambda^2} + \frac{1}{2}g^{\sigma\alpha}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0,$$

(rem: $g^{\alpha\beta} := (g^{-1})^{\alpha\beta}\,, \quad g^{\sigma\alpha}g_{\alpha\beta} = \delta^\sigma_\beta$)

$$\frac{d^2x^\sigma}{d\lambda^2} + \Gamma^\sigma_{\beta\gamma}(x)\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \quad \textbf{(geodesic equation!)}$$