"ratio criterium"

**Theorem:** $\sum_{n=1}^{\infty} a_n \quad \boxed{a_n > 0} \quad \forall n \in \mathbb{N}$

1.) if $\exists \boxed{\ell < 1}$ s.t. $\frac{a_{n+1}}{a_n} < \ell \quad \forall n \in \mathbb{N}$ (or $n \geq \bar{n}$)

then the series $\sum_{n=1}^{\infty} a_n$ converges

2.) if $\frac{a_{n+1}}{a_n} \geq 1 \quad \forall n \geq \bar{n}$ (for some $\bar{n}$)

then the series diverges

**Proof of 1.):**

$$\frac{a_{n+1}}{a_n} < \ell \iff a_{n+1} < \ell a_n$$

also $a_n < \ell a_{n-1}$

$$\implies a_{n+1} < \ell^2 a_{n-1}$$

$$a_{n+1} < \ell^3 a_{n-2}$$

$$a_{n+1} < \ell^n a_1 \quad \forall n \in \mathbb{N}$$

$$\Updownarrow$$

$$a_n < \ell^{n-1} a_1 \quad (*)$$

$d.\ \sum_{n=1}^{\infty} \ell^n \leftarrow$ geometric series with base $\ell < 1$

$\Downarrow$

it converges.

by comparison applied $(*)$ $\Longrightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

Proof of 2)

$$\frac{a_{n+1}}{a_n} \geq 1 \qquad \forall n > \bar{n}$$

$$\Updownarrow$$

$$a_{n+1} \geq a_n \geq a_{n-1} \cdots\cdots \geq a_1 > 0 \qquad \forall n > \bar{n}$$

$$\Longrightarrow \lim_{n \to +\infty} a_n \neq 0$$

so the series does not converge $\Longrightarrow$ it diverges.

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Corollary (asymptotic ratio criterion)

If $\sum a_n \qquad a_n > 0 \quad (n > \bar{n})$

- 1) $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \ell < 1$

$$\Longrightarrow \sum a_n \text{ converges.}$$

- 2) $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \ell > 1$

then the series $\sum a_n$ diverges.

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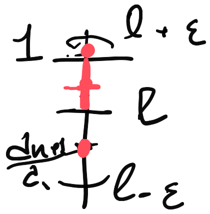
Proof of 1):

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \ell < 1$$



there exists $\bar{n}$ s.t. $\forall n > \bar{n}$ $l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon$

$\frac{a_{n+1}}{a_n} < \bar{l} < 1$ for some $\bar{l}$

Choice: $\varepsilon = \frac{1-l}{2}$ $\bar{l}$,

$$l - \frac{1-l}{2} < \frac{a_{n+1}}{a_n} < l + \frac{1-l}{2} = \boxed{\frac{l+1}{2}} < 1$$

Apply Theorem and get the thesis.

Proof of 2) by exercise.

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We proved Example 1

$\sum \frac{n^n}{(2n)!}$ is convergent by ratio criterium

**Gift**

$\lim \frac{n^n}{(2n)!} = 0$

Example 2:

$$\boxed{\sum \frac{n^n}{n!}}$$

ratio criterium

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \Big/ \frac{n^n}{n!} =$$

$$\frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} =$$

$$\cancel{(n+1)} \underbrace{\boxed{\left(\frac{n+1}{n}\right)^n}}_{\to e} \cdot \frac{1}{\cancel{n+1}} \to e > 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{n!} = +\infty$$

We knew that

$\sum \frac{n^n}{n!}$ does not converge

also by

$$\frac{n^n}{n!} = \frac{\overbrace{n \cdot n \cdot n \ldots n}^{n \text{ times}}}{n \cdot (n-1) \cdot (n-2) \ldots 1} =$$

$$= \underbrace{\frac{n}{n}}_{\to 1} \cdot \underbrace{\frac{n}{n-1}}_{\to 1} \cdot \underbrace{\frac{n}{n-2}}_{\to 1} \cdots \cdots \frac{n}{1} = +\infty$$

the necessary cond. $\lim a_n = 0$ is violated.

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$$\sum_{n=0}^{\infty} \frac{2^n}{\cosh(n)}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\cancel{2^{n+1}}\ 2}{\frac{e^{n+1} + e^{-(n+1)}}{\cancel{2}}} \cdot \frac{\frac{e^n + e^{-n}}{\cancel{2}}}{\cancel{2^n}} =$$

$$2 \lim_{n \to +\infty} \frac{\cancel{e^n}(1 + \overset{\to 1}{e^{-2n}})}{e \cancel{e^{n+1}}(1 + \underset{\to 1}{e^{-2n-2}})} = \frac{2}{e} < 1$$

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What can we conclude if $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = 1$? NOTHING!

Indeed consider

$\boxed{\sum \frac{1}{n} \text{ does not conv}}$ $\boxed{\sum \frac{1}{n^2} \text{ converges}}$

Apply ratio crit:

For $\sum \frac{1}{n}$ ... $\frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$

For $\sum \frac{1}{n^2}$ $\frac{\frac{1}{(n+1)^2}}{\frac{1}{(n)^2}} = 1$

**Theorem 2** ("nth root criterium"). $\sum a_n$, $a_n > 0$

1) $\sqrt[n]{a_n} < l < 1 \quad \forall n > \bar{n}$
then $\sum a_n$ converges

2) $\sqrt[n]{a_n} \geq 1 \quad \forall n > \bar{n}$
then the series diverges

**Corollary**

1) $\lim \sqrt[n]{a_n} = l < 1$
the series converges

2) $\lim \sqrt[n]{a_n} = l > 1$
the series diverges

(if $\lim \sqrt[n]{a_n} = 1$ nothing can be deduced)

**Proof of 1) of Theorem 2:**

by hypothesis $\sqrt[n]{a_n} < l < 1$

$a_n < l^n$

$\sum l^n$ geom series converges because $l<1$

by comparison

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

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**Example**

$$\sum \frac{\left(1 - \log\left(1+\frac{1}{n}\right) - \frac{1}{n}\right)^n}{2\left(1+\frac{\alpha}{n}\right)^{n^2}} \qquad \alpha \geq 0$$

positive terms

For which $\alpha$ does it converge?

n-th root crit:

$$\sqrt[n]{\frac{\left(1 - \log\left(1+\frac{1}{n}\right) - \frac{1}{n}\right)^n}{2\left(1+\frac{\alpha}{n}\right)^{n^2}}} =$$

$$\frac{\frac{1}{\sqrt[n]{2}}}{} \cdot \frac{1-\log\left(1+\frac{1}{n}\right)-\frac{1}{n}}{\left(1+\frac{\alpha}{n}\right)^{n}} = \frac{1}{e^{\alpha}}$$

$1 \quad\quad 1$

$$\frac{1}{e^{\alpha}} < 1 \iff e^{\alpha} > 1 \iff \boxed{\alpha > 0}$$

$$\boxed{\lim_{n\to+\infty}\left(1+\frac{\alpha}{n}\right)^{n}} \quad \alpha > 0$$

~~try~~

$$\boxed{\lim_{x\to+\infty}\left(1+\frac{\alpha}{x}\right)^{x}} = \quad \boxed{y=\frac{x}{\alpha}}$$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y\alpha} = \lim_{y\to+\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{\alpha} =$$

$= e^{\alpha}$. Hence:

For every $\alpha > 0$ the series converges.

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If $\alpha = 0$ $\quad \sum \left(\frac{1-\log\left(1+\frac{1}{n}\right)-\frac{1}{n}}{2}\right)^{n}$

Check $\lim a_n = 0$?

$$\lim \frac{1}{2}\left(1-\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\right)^{n} =$$

$$\frac{1}{2}\cdot\frac{1}{e} = \frac{1}{2e} \neq 0$$

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$\sum (-1)^n \frac{1}{n}$ $\quad$ $\boxed{\sum (-1)^n \frac{1}{n^2}}$

This does not converge absolutely BUT I KNOW THAT IT CONVERGES

It converges absolutely:

$$\sum \left|(-1)^n \frac{1}{n^2}\right| = \sum \frac{1}{n^2}$$

it converges!

$\Longrightarrow \sum (-1)^n \frac{1}{n^2}$ converges absolutely

$\Longrightarrow$ it converges

**Theorem** (Leibniz criterium for series with alternating sign)

$$\sum_{n=1}^{\infty} \underbrace{(-1)^n b_n}_{a_n} \qquad b_n \geq 0$$

Assume that:

- $\lim_{n \to +\infty} b_n = 0$
- $(b_n)_{n \in \mathbb{N}}$ is decreasing

Then $\sum_{n=1}^{\infty} (-1) b_n$ converges

Examples:

$\sum (-1) \frac{1}{n^\alpha}$ $\quad \forall \alpha > 0$

- $\lim_{n \to +\infty} \frac{1}{n^\alpha} = 0$
- $\frac{1}{n^\alpha}$ decreasing

$\Longrightarrow$ it converges $\forall \alpha > 0$ .)

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Example.

$$\sum_{n=0}^{\infty} \underbrace{\sin\left(\frac{\pi}{2} + n\pi\right)}\cdot \frac{\frac{1}{\sqrt{n}}}{(n^2+2n)^\alpha} \qquad \alpha \geq 0$$

Study convergenc and abs. convergence:

$$\sum = \sum (-1)^n \frac{\frac{1}{\sqrt{n}}}{(n^2+2n)^\alpha}$$

Absol. conv.

$$\sum_{n=1}^{\infty} \frac{\frac{1}{\sqrt{n}}}{(n^2+2n)^\alpha}$$

$$\frac{\frac{1}{\sqrt{n}}}{(n^2+2n)^\alpha} \sim \frac{1}{\sqrt{n}(n^2+2n)^\alpha} = \frac{1}{n^{\frac{1}{2}} n^{2\alpha}} = \frac{1}{n^{(\frac{1}{2}+2\alpha)}}$$

Converges $\Longleftrightarrow \sum \frac{1}{n^{(\frac{1}{2}+2\alpha)}}$ conv.

$\Longleftrightarrow \frac{1}{2} + 2\alpha > 1$

$\Updownarrow$

$2\alpha > \frac{1}{2} \Longleftrightarrow \alpha > \frac{1}{4}$

$\uparrow$

absbos. conv. $\Longrightarrow$ conv.

What happens for $\boxed{0 \leq \alpha \leq \frac{1}{4}}$

No abs. conv.

Try Leibniz crit.

$$\sum (-1)^n \underbrace{\frac{1}{\sqrt{n}\,(n^2+2n)^\alpha}}_{b_n}$$

- $\lim_{n\to+\infty} b_n = 0$ ? Yes!

- is $\frac{1}{\sqrt{n}\,(n^2+2n)^\alpha}$ decreasing

$\boxed{\text{directly: yes}}$

or prove that

$$\frac{d\left(\frac{1}{\sqrt{x}\,(x^2+2x)^\alpha}\right)}{dx} \leq 0$$

by Leibniz $\Longrightarrow$ The series is converging also for $0 \leq \alpha \leq \frac{1}{4}$