- $\sum_{n=0}^{+\infty} a_n$ it can converge to $S \in \mathbb{R}$
  diverge to $\pm\infty$
  indeterminate
- Theorem 1: If $\sum_{n=1}^{\infty} a_n$ converges
  then $\lim a_n = 0$
- Particular series
  $\sum_{n=0}^{+\infty} q^n$ ("geometric") $(q > 0)$
  converges $\Leftrightarrow 0 < q < 1$
  ($\Rightarrow$ diverges $\Leftrightarrow q \geq 1$)
- Mengoli's series
  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.
- $\sum \frac{1}{n^\alpha}$ ("harmonic" with exponent $\alpha > 0$)
  converges if and only if $\alpha > 1$

Series with terms ≥ 0

- $\sum_{n=0}^{\infty} a_n \qquad a_n \geq 0 \quad \forall n \in \mathbb{N}$

converges or diverges.

Theorem 2 (Comparison)

$0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} b_n < +\infty \Longrightarrow \sum_{n=1}^{\infty} a_n < +\infty$$

$$\left(\Downarrow \text{ if } \sum_{n=1}^{\infty} a_n = +\infty \Longrightarrow \sum_{n=1}^{\infty} b_n = +\infty \right).$$

- Theorem 3 $\quad f : [0, +\infty[\to \mathbb{R}$

- f is decreasing
- $\lim_{x \to +\infty} f(x) = 0$

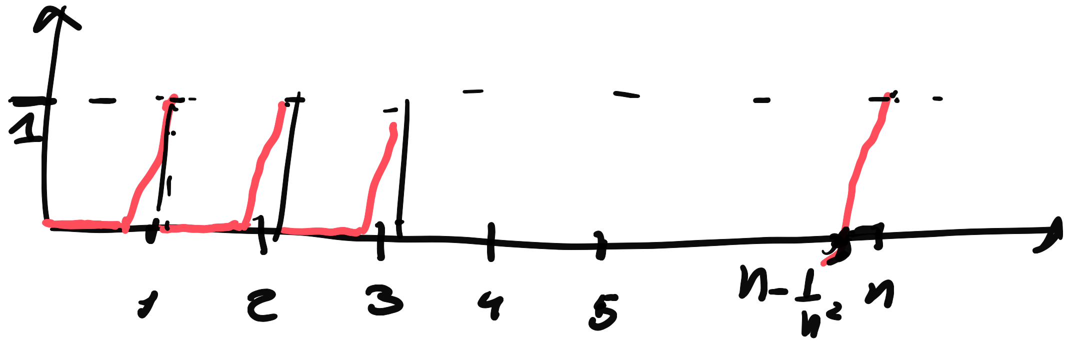
$$\sum_{n=1}^{+\infty} f(n) < +\infty \Longleftrightarrow \int_1^{+\infty} f(x)\,dx < 0$$

Curiosity: You might think that as in the case of series ($\sum a_n$ conv. $\Rightarrow \lim a_n = 0$) that if $\int_1^{+\infty} f(x)\,dx$ converges

then $\lim_{x\to+\infty} f(x) = 0$

FALSE!

Counterexample



$n-1 \le x \le n$

$$f(x) = \begin{cases} 0 & n-1 < x < n-\frac{1}{n^2} \\ n^2(n-x) & n-\frac{1}{n^2} \le x \le n \end{cases}$$

$$\int_0^{1} f(x) = \sum_{n=1}^{+\infty} \left(\frac{1}{n^2}\cdot 1\right)\frac{1}{2} =$$

$$= \frac{1}{2}\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$$

$\lim_{x\to+\infty} f(x) = 0$? No

(asymptotic comparison 1)

Theorem 4: $a_n \geq 0$ $b_n \geq 0$

$a_n = o(b_n)$

If $\sum_{n=1}^{\infty} b_n$ converges

then $\sum_{n=1}^{\infty} a_n$ converges

Proof: $a_n = o(b_n) \iff \lim_{n\to\infty} \frac{a_n}{b_n} = 0$

$\iff \forall \varepsilon > 0 \quad \exists \bar{n} \text{ s.t. } \forall n > \bar{n}$

$$-\varepsilon < \frac{a_n}{b_n} < \varepsilon$$

if $\varepsilon = 1$ $\exists \tilde{n}$: $\forall n \geq \tilde{n}$

$$\frac{a_n}{b_n} < 1$$

$$a_n < b_n \quad \forall n \geq \tilde{n}$$

$\Downarrow$ comparison

$\sum a_n$ converges.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{\log\left(1+\frac{1}{n}\right)}{n^{\gamma}}$

For which γ this series converges

I know that $\log\left(1+\frac{1}{n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right)$

$$\frac{\log\left(1+\frac{1}{n}\right)}{n^\gamma} = \frac{\frac{1}{n}+o\left(\frac{1}{n}\right)}{n^\gamma} =$$

$$= \frac{1}{n^{\gamma+1}} + o\left(\frac{1}{n^{\gamma+1}}\right) =$$

$$o\left(\frac{1}{n^\alpha}\right)$$

provided $\gamma+1>\alpha$

Apply asymptotic th. n.1

But $\sum \frac{1}{n^\alpha}$ converges for $\alpha>1$

So:

if $\gamma+1>\alpha>1$

our series converges

$$\gamma+1>1$$

$$\iff \gamma>0$$

<u>Theorem 5</u> (asymptotic n.2)

hypothesis:

$a_n>0 \quad b_n>0 \quad \forall n\in\mathbb{N}$

$a_n \sim b_n$ ("a_n is asymptotic to b_n")

$\Updownarrow$

$$\lim \frac{a_n}{b_n} = l > 0$$

Thesis

$\sum a_n$ converges

$\boxed{\Downarrow}$

$\sum b_n$ converges

Proof: $\forall \varepsilon > 0 \ \exists \bar{n}$ s.t.

$\forall n \geq \bar{n} \quad l - \varepsilon < \dfrac{a_n}{b_n} < l + \varepsilon$

a "smart" choice of ε, is an $\varepsilon < l$, e.g., $\varepsilon = \dfrac{l}{2}$

$\Rightarrow \quad \dfrac{l}{2} < \dfrac{a_n}{b_n} < \dfrac{3}{2} l$

$\forall n > \bar{n} \quad$ (I) (II)

From (I) $\Rightarrow b_n \dfrac{l}{2} < a_n \quad (*)$

$\forall n > \bar{n}$

From (II) $\Rightarrow a_n < \dfrac{3}{2} l \, b_n \quad (**)$

From $(*)$ $\sum a_n < +\infty \Rightarrow \sum b_n \cdot \dfrac{l}{2} < +\infty$

$\Leftrightarrow \sum b_n < +\infty$

Also

From $(**)$, if $\sum_{n=1}^{+\infty} b_n < +\infty$

$\Leftrightarrow \sum_{n=1}^{+\infty} \dfrac{3}{2} l \, b_n$ converges

$$\underset{\text{comparison}}{\Longrightarrow} \quad \sum a_n < +\infty$$

q.e.d.

<u>Example</u>: again

$$\sum \frac{\log\left(1+\frac{1}{n}\right)}{n^\gamma}$$

$$\frac{\log\left(1+\frac{1}{n}\right)}{n^\gamma} = \frac{\frac{1}{n} + o\left(\frac{1}{n}\right)}{n^\gamma}$$

$$\sim \frac{1}{n^{\gamma+1}}$$

$$\left(\text{indeed} \quad \lim_{n\to+\infty} \frac{\left(\frac{\frac{1}{n} + o\left(\frac{1}{n}\right)}{n^\gamma}\right)}{\frac{1}{n^{\gamma+1}}} = 1\right)$$

i.e. $$\frac{\log\left(1+\frac{1}{n}\right)}{n^\gamma} \sim \frac{1}{n^{\gamma+1}}$$

by asymp. compariso n.e

$$\sum \frac{1+\frac{1}{n}}{n^\gamma} < +\infty \Longleftrightarrow \sum \frac{1}{n^{\gamma+1}} < +\infty$$

$$\iff \gamma+1>1 \iff \gamma>0.$$

Exercise

$$\sum_{n=1}^{+\infty} \frac{\sinh\left(\frac{1}{n^2+n^3}\right)}{\operatorname{arctg}(n^{-2\alpha})}$$

$\alpha>0$. For which α the series converges?

$$\frac{\sinh\left(\frac{1}{n^2+n^3}\right)}{\operatorname{arctg}\left(\frac{1}{n^{2\alpha}}\right)} = \frac{\frac{1}{n^2+n^3} + o\left(\frac{1}{n^2+n^3}\right)}{\frac{1}{n^{2\alpha}}} ;$$

$\sinh(x) = x + o(x) \quad x\to 0$

$\operatorname{arctg}(x) = x + o(x) \quad x\to 0$

$$= \frac{n^{2\alpha}}{n^2+n^3} + o\left(\frac{n^{2\alpha}}{n^2+n^3}\right)$$

$$\frac{n^{2\alpha}}{n^3\underbrace{\left(\frac{1}{n}+1\right)}_{\to 1}} \sim \boxed{n^{2\alpha-3}}$$

generic form $\sim n^{2\alpha-3}$ $\boxed{= \frac{1}{n^{3-2\alpha}}}$
Apply asymt. comp. 2.
Since $\sum \frac{1}{n^{3-2\alpha}} < +\infty$

$\Updownarrow$

$3-2\alpha > 1$

$\Updownarrow$

$-2\alpha > -2$

$\alpha < \underline{1}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \dots$$

Does it <u>converge absolutely</u>?

$\Updownarrow$

DOES $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

converge? <u>NO</u>

Can we conclude that

$\sum_{n=1} \frac{(-1)^n}{n}$ does not converge?

NO

Actually it converges.

Exercise

$$\boxed{\sum \frac{n^n}{(2n)!}} \qquad (\star)$$

Theorem ("ratio criterium")

$\sum_{n=1}^{\infty} a_n \qquad a_n > 0 \quad \forall n \in \mathbb{N}$

There exists $0 \leq l < 1$

s.t. $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = l \ (< 1)$

Then the series $\sum a_n$ converges.

Apply to $(\star)$:

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \to +\infty} \frac{\frac{(n+1)^{n+1}}{(2(n+1))!}}{\frac{n^n}{(2n)!}} =$$

$$\lim \frac{(n+1)^n (n+1)}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{n^n} =$$

$$\underbrace{\lim\left(1+\frac{1}{n}\right)^n}_{\downarrow\ e} \quad \underbrace{\boxed{\frac{n+1}{(2n+2)(2n+1)}}}_{\downarrow\ 0} = 0 < 1 .$$

by ration criterion

$$\sum \frac{n^n}{(2n)!} \quad \text{converges}$$

(by necessary condition)

$$\lim_{n\to+\infty} \frac{n^n}{(2n)!} = 0$$