$(a_n)_{n \in \mathbb{N}}$

partial sum

Series $\sum_{n=0}^{+\infty} a_n = \left\{ S_k = a_0 + \dots + a_k,\ k \in \mathbb{N} \right\}$

If $\lim_{n \to +\infty} S_k = S \in \mathbb{R}$ we say that the series "converges to S" or "the sum of the series is S"

and we write

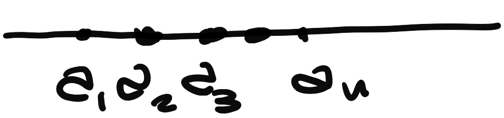
$$\sum_{n=0}^{+\infty} a_n = S$$

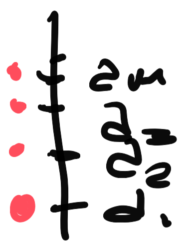
Interpretation of a series as a gener. integral..

Recall

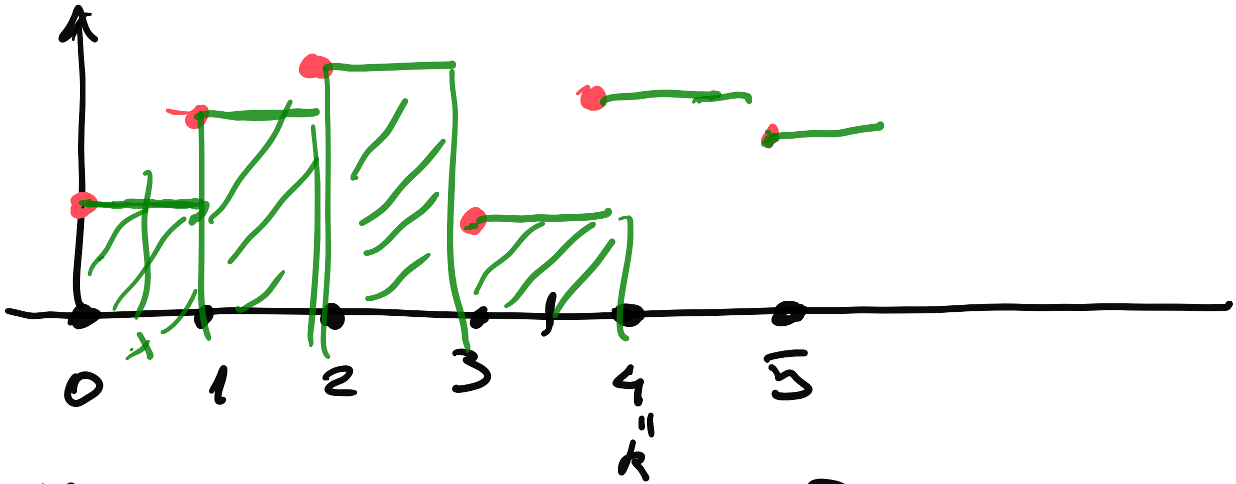
a sequence is

$(a_n)_{n \in \mathbb{N}}$





A sequence is a function defined on $\mathbb{N}$



$$\sum_{n=0}^{+\infty} a_n = \{ S_k = a_0 + a_1 + \dots a_k \}$$

$([x] :=$ integer part of $x)$

Consider

$$f(x) = a_{[x]}$$

$$\int_0^k f(x)\,dx = a_0 + a_1 + \quad a_{k-1} = S_{k-1}$$

$$\xrightarrow{k \to +\infty} \qquad \xrightarrow{k \to \infty}$$

$$\int_0^{+\infty} f(x)\,dx = \sum_{k=0}^{\infty} a_n = S.$$

Particularly important Series

$$\sum_{n=0}^{+\infty} q^n$$

Geometric Series of "ration" q. $q > 0$

• $q = 1$

$S_k = (k+1)1 = k+1 \to +\infty$

the series diverges

$\sum_{n=0}^{\infty} q^n = +\infty$

• $q > 1$

$S_k = q^0 + q + q^2 + \dots$

$S_k \to +\infty$

(by applying Th. $(a_n)(b_n)$ $0 \le a_n \le b_n$

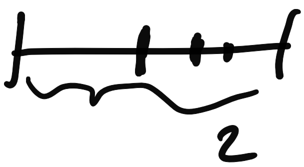
if $\sum_{n=0}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=0} b_n$ diverges)

$q^n > 1$

• $q < 1$

$S_k = \underbrace{1 + q + q^2 + \dots + q^k}$

$q = \frac{1}{2}$ $\quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots$



$$\underbrace{(1+q+q^2+\dots+q^k)}_{S_k}(1-q) =$$

$$= 1+\cancel{q}+\cancel{q^2}+\dots+\cancel{q^k}$$

$$-\cancel{q}-\cancel{q^2}\dots-\cancel{q^k}-q^{k+1} =$$

$$= 1 - q^{k+1}$$

$q \neq 1$

$$S_k = \frac{1-q^{k+1}}{1-q}$$

$k \to \infty$

$+\infty$ if $q>1$

$\frac{1}{1-q}$ $\boxed{\text{if } q<1}$

<u>Particular case</u>

$$q=\frac{1}{2} \quad \frac{1}{1-q} = \frac{1}{\frac{1}{2}} = 2$$

<u>Theorem</u> if $(a_n)_{n\in\mathbb{N}}$ is such that $a_n \geq 0$ $\forall n\in\mathbb{N}$ then the either the series converges or " " diverges

<u>Proof</u>: the sequence

$$\overset{a_0}{S_0}, \overset{a_0+a_1}{S_1}, \overset{a_0+a_1+a_2}{S_2}, \dots S_n \dots$$

is increasing

$\Longrightarrow$ it has a limit, finite or infinite,

i.e. $\sum_{n=0}^{\infty} a_n$ either converges or diverges.

Theorem: $0 \leq a_n \leq b_n \ \forall n \in \mathbb{N}$

Then

1) if $\sum_{n=0}^{\infty} a_n$ diverges $\Longrightarrow$ $\boxed{\sum_{n=0}^{\infty} b_n \text{ diverges}}$

2) If $\sum_{n=0}^{+\infty} b_n$ converges $\Longrightarrow \sum_{n=0}^{\infty} a_n$ converges.

Proof Proof of 1)

$S_k^a := a_0 + \dots + a_k$

$S_k^b := b_0 + b_1 \dots b_k$

$S_k^a \leq S_k^b$

by hypothesis $S_k^a \to +\infty$

compar. for seq.

$\Longrightarrow S_k^b \to +\infty$

i.e. $\sum_{n=0} b_n$ diverges.

2) is already proved

(because $P \Rightarrow Q \Leftrightarrow \text{non}\,Q \Rightarrow \text{non}\,P$)

The "harmonic" series

$$\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

The "generalized harmonic series of exponent $\alpha > 0$"

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^\alpha}} = \begin{cases} \alpha \le 1 & \underline{\text{diverges}} \\ \alpha > 1 & \underline{\text{converges}} \end{cases}$$

(Remember?

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \begin{cases} < +\infty & \alpha > 1 \\ = +\infty & \alpha \le 1 \end{cases}$$
)

$a_1 + a_2 + a_3 + \dots + a_k$

(a_n) $\qquad \sum a_n$

the fact that $a_n \xrightarrow[n\to+\infty]{} 0$

is not sufficient

in order that $\sum_{n=0}^{\infty} a_n$ is convergent

Indeed $\frac{1}{n^\alpha} \longrightarrow 0$

for both $\alpha \leq 1$ and $\alpha > 1$

is the following theorem true?

Theorem: If $\sum_{n=0}^{+\infty} a_n = S \in \mathbb{R}$ converges

$\Downarrow$

$\lim_{n\to+\infty} a_n = 0$

$\boxed{\lim_{n\to+\infty} a_n = 4}$ $\quad \forall \underline{n > N}$

$4-1 < a_n < 4+1$

$3 < a_n < 5$

$S_k = a_0 + \dots + a_N + \underbrace{\overset{3}{\widehat{a_{N+1}}} \dots + \overset{3}{\widehat{a_k}}}$

Proof $a_n = S_n - S_{n-1} = $

$= \cancel{a_0} + \cancel{a_1} \cdots \cancel{a_{n-1}} + a_n$

$- \cancel{a_0} - \cancel{a_1} \cdots - \cancel{a_{n-1}}$

$$\lim_{n \to +\infty} a_n = \underbrace{\lim_{n \to +\infty} S_n}_{S} - \underbrace{\lim_{n \to \infty} S_{n-1}}_{S} =$$

$$= S - S = 0.$$

(Mengoli's Series)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges?
diverges?

$$S_k = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{k(k-1)} =$$

$$= \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) =$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{\cancel{n}+1-\cancel{n}}{n(n+1)}$$

$$\hookrightarrow \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots$$

$$+ \left(\frac{1}{k} - \frac{1}{k+1}\right) =$$

$$S_k = \boxed{1 - \frac{1}{k+1}} \xrightarrow[k \to +\infty]{} 1$$

Observe that

$$\underbrace{\boxed{\frac{1}{(n+1)^2}}}_{\substack{(n+1)\text{-th} \\ \text{term of} \\ \text{harmonic} \\ \alpha = 2}} \leq \underbrace{\boxed{\frac{1}{n(n+1)}}}_{\substack{n\text{-th} \\ \text{term} \\ \text{of Mengoli}}}$$

$\Downarrow$ comparison

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges

If $\alpha > 2$

$$\frac{1}{n^\alpha} < \frac{1}{n^2}$$

$$\xrightarrow[\substack{\text{comparison} \\ \text{with } \sum \frac{1}{n^2}}]{} \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \boxed{\alpha \geq 2}$$

converges

If we know that

$$\boxed{\sum \frac{1}{n} \quad \text{diverges}}$$

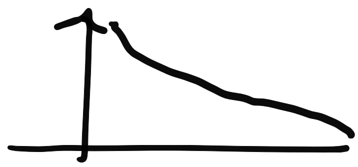
by comparison $\left(\frac{1}{n^\alpha} > \frac{1}{n} \text{ if } \alpha < 1\right)$

$$\sum \frac{1}{n^\alpha} \quad \text{diverges } \alpha \le 1$$

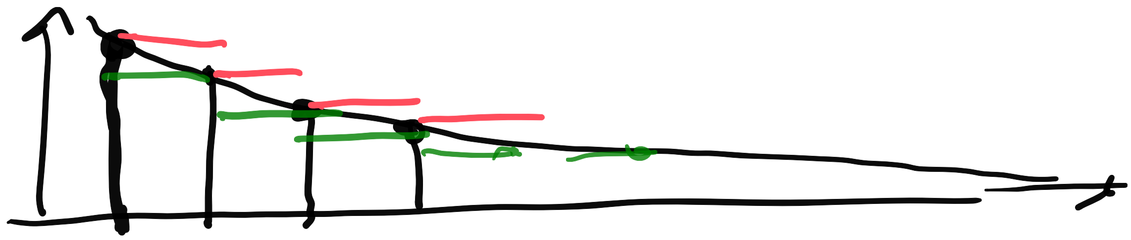
Theorem 3: $f : [0, +\infty[\to \mathbb{R}$

$f(x) \geq 0 \ \forall x$, f decreasin

$\lim\limits_{x \to +\infty} f(x) = 0$



$$\sum_{n=0}^{+\infty} f(n) < +\infty \iff \int_0^{+\infty} f(x)\,dx < +\infty$$



Apply Theorem 3 to harmonic of expont α

$$\sum \frac{1}{n^\alpha}$$

Choose $f = \frac{1}{x^\alpha}$ $\alpha > 0$

$f(n) = \frac{1}{n^\alpha}$

$f = \frac{1}{x^\alpha}$ is decreasing

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x^\alpha} = 0$

$\Downarrow$ Theorem 3

$\sum f(n) = \sum \frac{1}{n^\alpha}$

converges if and only if

$\alpha > 1$

(because $\int \frac{1}{x^\alpha} < +\infty \iff \alpha > 1$)