Happy birthday
Ludwig van
Beethoven!

Theorem 1 (Comparison)

$f, g : [a, b[\longrightarrow \mathbb{R}$ $\quad (b \in \mathbb{R} \cup \{+\infty\})$

$0 \leq f(x) \leq g(x) \quad \forall x \in [a, b[$

(f, g are Riemann-integrable on every $[a, c]$, $c < b$)

g integrable in general. sense $\Longrightarrow$ f integrable in gen. sense.

Remark. We can weaken the hypothesis by assumin

$\boxed{0 \leq f(x) \leq g(x)}$ $\forall x \in [d, b[$ for some $a \leq d \leq b$

Remark Similarly we have analogous results

$f, g :]a, b] \longrightarrow \mathbb{R}$

$a \in \mathbb{R} \cup \{-\infty\}$

Theorem 2 (Asymptotic comparison)

$f, g : [a, b[\longrightarrow \mathbb{R}$, $f(x) \geq 0$, $g(x) \geq 0$

$f = o(g)$ as $x \to b^-$

If g is int. in gen. sense

then f is " " "

Proof $\lim_{x \to b^-} \frac{f(x)}{g(x)} = 0$

Case $b = +\infty$

$\forall \varepsilon > 0 \quad \exists M$ s.t. $-\varepsilon \le \frac{f(x)}{g(x)} \le \varepsilon$

$\forall x > M$

$f(x) \le \varepsilon g(x) \quad \forall x > M$

g is int. $\Rightarrow \varepsilon g(x)$ is int. gen. sense

$\Rightarrow$ $f(x)$ is int. " "

by Theorem 1

Exercise: Study integrability

$$\int_1^{+\infty} \frac{\log x}{x^3}\,dx$$

$$\underbrace{\frac{\log x}{x^3}} \leq \underbrace{\frac{K}{x^3}} \qquad \forall x \in [1,+\infty[$$

$$\frac{\log x}{x^3} = o\left(\frac{1}{x^{2.5}}\right) \; ?$$

$$\Updownarrow \qquad \qquad \uparrow$$

$$\lim_{x\to+\infty} \frac{\frac{\log x}{x^3}}{\frac{1}{x^{2.5}}} = \lim_{x\to+\infty} \frac{1}{x^{\frac{1}{2}}} \log(x) = 0$$

by Th.2 $\Rightarrow \displaystyle\int_1^{+\infty} \frac{\log(x)}{x^3}\,dx < +\infty$

I could have chosen $x^{2.999}$

$$3 - 2.999 = \frac{1}{1000}$$

$$\ldots\ldots \int_1^{+\infty} \frac{\log(x)}{x^{\frac{1}{1000}}} \to 0$$

Also we could have chosen x^α, with $\alpha < 3$

Theorem 3: $f, g : [a, b[\to \mathbb{R}$ and

$f, g \geq 0$ $g > 0$

f asymptotic to g for $x \to b$.

Then

f in int. in gen sense $\iff$ g is integ in gen sense

$$\left(\text{i.e. } \int_a^b f\,dx < +\infty \iff \int_a^b g\,dx \right)$$

Proof: By hyp.

$$\lim_{x \to b} \frac{f(x)}{g(x)} = L > 0$$

Let us take $\varepsilon = \frac{L}{2}$

$\exists M$ s.t. for every $x \geq M$

$$\boxed{\frac{L}{2} = L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon = \frac{3L}{2}}$$

Hence, $\forall x \geq M$,

$$f(x) \leq g(x) \cdot \frac{3}{2}L$$

If g is int. gen. sense

$\Updownarrow$

$\frac{3}{2}L\, g$ " " " "

Apply Theorem 1

$\Longrightarrow$ f is int. in gen. sense

(We have proved "$\Longleftarrow$")

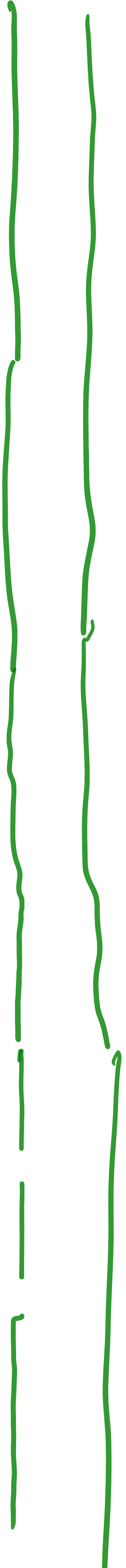
Let us prove "$\Longrightarrow$"

$$\frac{L}{2} g(x) \leq f(x)$$

$\forall x > M$, by compar. Th. 1

$\frac{L}{2} g(x)$ is int. in gen. sense

$\Longrightarrow g(x) = \frac{2}{L} \cdot \frac{L}{2} g(x)$ is int. in gen. sense



$$\int_1^{+\infty} \alpha \sin\left(\frac{1}{x}\right) - \alpha^2 \operatorname{arctg}\left(\log\left(1+\frac{1}{x}\right)\right) dx$$

For which $\alpha \in \mathbb{R}$ do we have absolute integrability?

Idea: - No problems at the extreme 1.

- Observe that $\frac{1}{x} \to 0+$ when $x \to +\infty$

$$\sin\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)$$

$$\boxed{\operatorname{arctg}\left(\log\left(1+\frac{1}{x}\right)\right)} = \log\left(1+\frac{1}{x}\right) - \frac{\left(\log\left(1+\frac{1}{x}\right)\right)^3}{3} + o\left(\left(\log\left(1+\frac{1}{x}\right)\right)^3\right) = ⊛$$

near $y=0$

$$\operatorname{arctg} y = y - \frac{2}{6}y^3 + o(y^3)$$

$$(\operatorname{arctg} y)'\Big|_0 = \frac{1}{1+y^2}\Big|_{y=0} = 1$$

$$(\operatorname{arctg} y)''\Big|_{y=0} = \left(\frac{1}{1+y^2}\right)'\Big|_{y=0} = -\frac{2y}{(1+y^2)^2}\Big|_{y=0}$$

$= 0$

$$\left(\operatorname{arctg}\right)'''_{y=0} = \left(-\frac{2y}{(1+y^2)^2}\right)'_{y=0} =$$

$$-\left.\frac{2(1+y^2)-2y(\ldots\ldots)}{(1+y^2)^4}\right|_{y=0}$$

$$= \frac{-2}{1} = -2$$

$$(*) = \frac{1}{x} - \frac{1}{2x^2} + \cancel{\frac{1}{3x^3}} + o\left(\frac{1}{x^3}\right) - \cancel{\frac{1}{3x^3}}$$

$$+ o\left(\frac{1}{x^3}\right) + o\left(\frac{1}{x^3}\right)$$

$$\int |f|\, dx =$$

$$\int_1^{+\infty} \alpha\left(\frac{1}{x} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)\right) - $$

$$- \alpha^2\left(\frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^3}\right)\right|$$

$$= \int_1^{+\infty} \left| (\alpha-\alpha^2)\frac{1}{x} + \frac{\alpha^2}{2x^2} + \alpha\left(o\left(\frac{1}{x^3}\right) - \frac{1}{6x^3}\right) - \alpha\, o\left(\frac{1}{x^3}\right)\right| dx$$

$$\boxed{|f| \sim \frac{1}{x}}$$

if $(\alpha-\alpha^2)\neq 0$

$$\lim_{x\to+\infty} \frac{\left|(\alpha-\alpha^2)\frac{1}{x} + o\left(\frac{1}{x}\right)\right|}{\frac{1}{x}} = |\alpha-\alpha^2|$$

Hence, since $\frac{1}{x}$ is not int. in gen. sense on $[1,+\infty[$

$$\alpha-\alpha^2\neq 0$$
$$\Updownarrow$$
$$\alpha(1-\alpha)\neq 0$$
$$\Updownarrow$$
$$\alpha\neq 1 \quad \alpha\neq 0$$

$\forall\ \alpha \in \mathbb{R}\setminus\{0,1\}$ the function is NOT integr. in gen. sense.

$\alpha = 0$ is trivial $f\equiv 0$

$\alpha = 1 \quad |f| \sim \dfrac{1}{2x^2}$

But $\dfrac{1}{2x^2}$ is integrable

on $[1, +\infty[$, i.e. $\displaystyle\int_1^{+\infty} \frac{1}{2x^2}\,dx < 0$

By theorem 3

$\Rightarrow |f|$ is integrable.

Let us consider

$(a_n)_{n\in\mathbb{N}}$

for instance

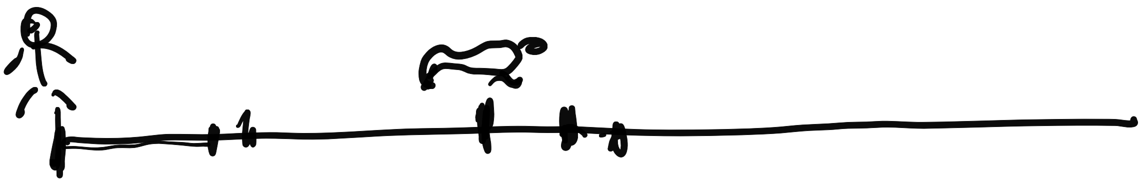
$a_n = \frac{1}{n}$

$a_n = 9^n$

"Sum"

$a_1 + a_2 + a_3 + \dots + a_{100} + \dots$

Zeno's paradox

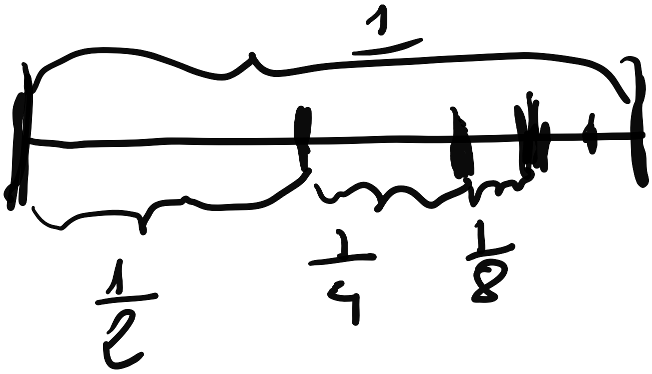


Achilles reaches the turtle at time

$t_1 + t_2 + t_3 \dots + t_n = +\infty$

paradox?

NO because the sum of infinite terms can be finite.



$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

k $\qquad$ " $\frac{7}{8}$

$a_1 = \frac{1}{2} = \frac{1}{2^1}$

$a_2 = \frac{1}{4} = \frac{1}{2^2}$ $\quad \dots \quad a_n = \left(\frac{1}{2}\right)^n$

$a_3 = \frac{1}{8} = \frac{1}{2^3}$

<u>Definition</u>. If (a_n) is a sequence, we call $\forall k \in \mathbb{N}$, "partial sum until k"

$$S_k := \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_n$$

The family of partial sums $\{S_k, k \in \mathbb{N}\}$ is called the SERIES associated to (a_n) and is denoted by $\sum_{n=1}^{\infty} a_n$

if $\lim_{k \to +\infty} S_k = S \in \mathbb{R}$

we say that "the series

has finite sum S"
or
" the series converges to S"

An example of non convergent
series:

$a_n = (-1)^n$

$S_1 = -1$

$S_2 = a_1 + a_2 = -1 + 1 = 0$

$S_3 = a_1 + a_2 + a_3 = -1$

$S_4 = 0$

$S_{2h+1} = 0$

$S_{2h} = -1$

$\Longrightarrow$ S_k does not
converge

$a_n = \left(\frac{1}{e}\right)^n$ $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent

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$a_n = \frac{1}{n}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ not convergent.