$$\int_a^b \frac{1}{(b-x)^\alpha} < +\infty \iff \alpha < 1$$

$$\int_0^1 \frac{1}{x^\alpha}\,dx < +\infty \iff \alpha < 1$$

$$\int_2^3 \frac{1}{(x-2)^\alpha} < +\infty \iff \alpha < 1$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx < +\infty \iff \alpha > 1$$

$$\int_2^{+\infty} \frac{1}{(x-2)^\alpha} < +\infty \iff \alpha > 1$$

(Language: "there $\exists$ gen. int" also "it converges")

$$\int_1^{+\infty} e^{\beta x}\,dx = \lim_{b\to+\infty} \int_1^b e^{\beta x}\,dx =$$

$\beta \neq 0$

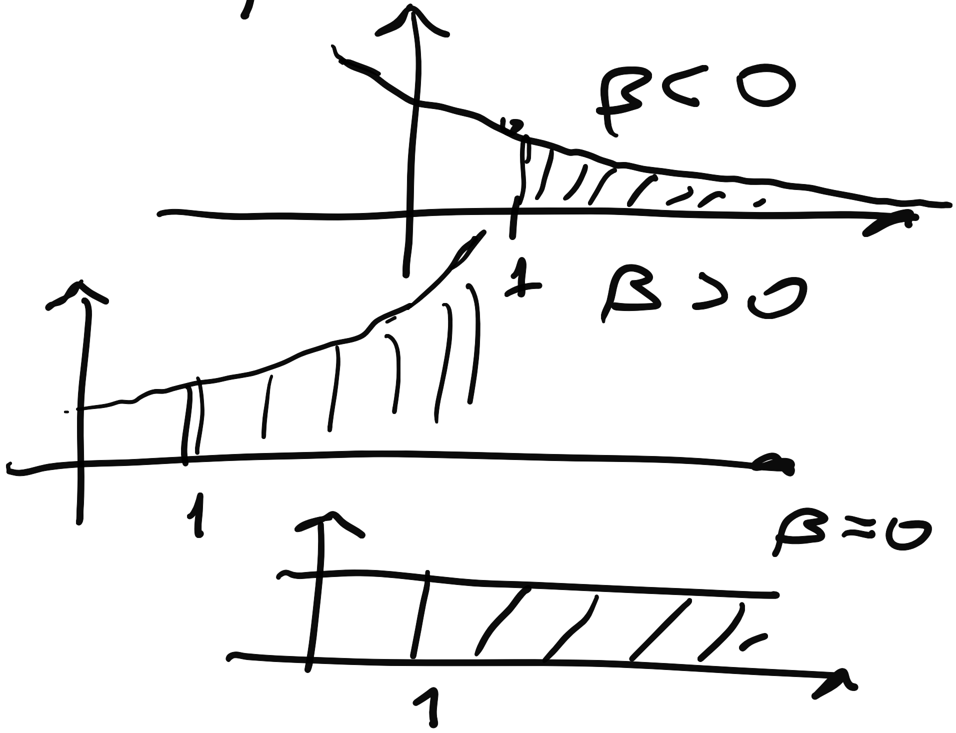
$$= \lim_{b\to+\infty} \frac{e^{\beta x}}{\beta}\Bigg|_1^b = \lim_{b\to+\infty} \frac{e^{\beta b}}{\beta} - \frac{e^{\beta}}{\beta}$$

$$= \begin{cases} +\infty & \text{if } \beta > 0 \\ -\dfrac{e^\beta}{\beta} & \text{if } \beta < 0 \end{cases}$$

converges $\iff \beta < 0$

(the case $\beta = 0$

$$\int_1^{+\infty} 1 \, dx = \lim_{b \to +\infty} (b-1) = +\infty \Big)$$



$$\int_a^b \quad \leftarrow \text{gen. integrals}$$

when one of a, b is not standard
that is, either a or b is $\pm\infty$
or the function is unbounded near a or b

<u>Def</u> $$\int_a^{+\infty} f(x)\, dx := \int_a^c f(x) + \int_c^{+\infty} f(x)\, dx \quad (*)$$

if $f \in \mathcal{R}([\bar{a}, \bar{b}])$ where $\bar{a} > a$, $\bar{b} < +\infty$

We say that f is int. in the gen. sense if for any $c \in]a,+\infty[$

$\int_a^c f(x)\,dx$ exists $\int_c^{+\infty} f(x)\,dx$ exists

and we set $(*)$.

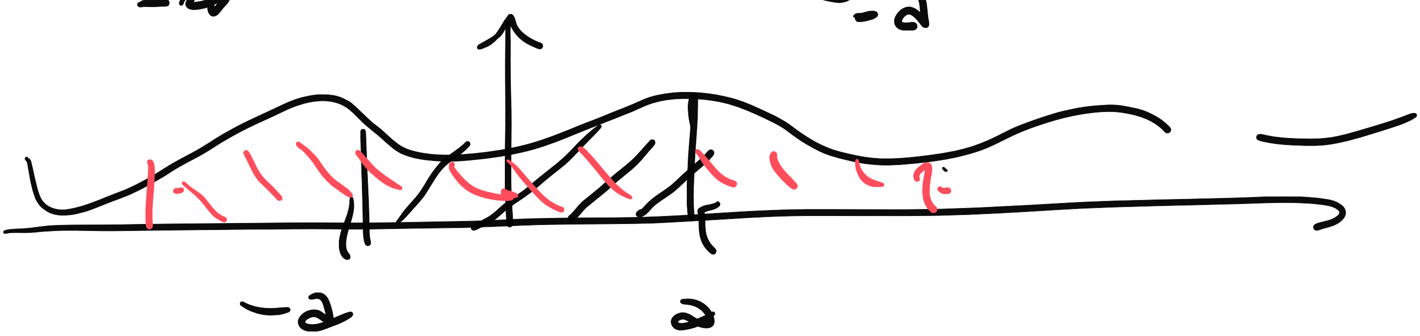
Also

$$\int_{-\infty}^{+\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

Why this strange definition?

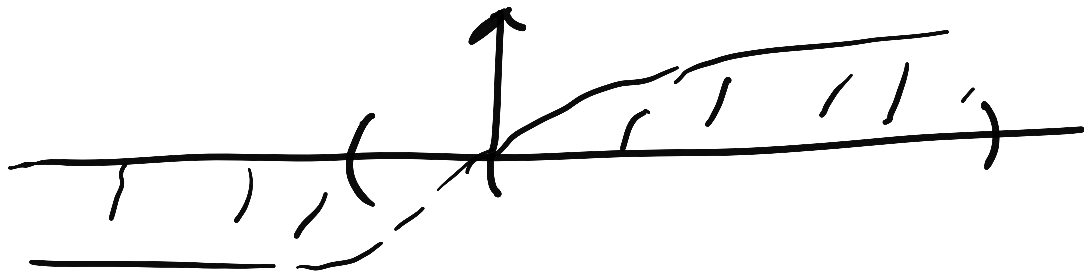
Why not

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to +\infty} \int_{-a}^{a} f(x)\,dx$$



$$\int_{-\infty}^{+\infty} \operatorname{arctg} x = \lim_{a \to +\infty} \underbrace{\int_{-a}^{+a} \operatorname{arctg} x}_{0} = 0$$

not the right definition.



Exercise: For which $\alpha \in \mathbb{R}$ the function $f(x) = \frac{1}{x^\alpha}$ $\quad f:]0, +\infty[$ is intg. in gen. sense?

$$\int_0^{+\infty} \frac{1}{x^\alpha} dx \underset{\substack{\text{if it}\\ \text{exists}}}{=} \int_0^1 \frac{1}{x^\alpha} dx + \int_1^{+\infty} \frac{1}{x^\alpha} dx$$

$\int_0^1 \frac{1}{x^\alpha} dx$ it exists $\iff \alpha < 1$

$\int_1^{+\infty} \frac{1}{x^\alpha} dx$ " " $\iff \alpha > 1$

No good α exists!

The defin. of

$$\int_a^{+\infty} f(x)\, dx = \int_a^c f(x)\, dx + \int_c^{+\infty} f(x)\, dx$$

does not depend on c:

Let $\bar{c} \neq c$

$$\boxed{\int_a^{\bar c} f(x)\,dx} = \lim_{\bar a \to a^+} \int_{\bar a}^{\bar c} f(x)\,dx =$$

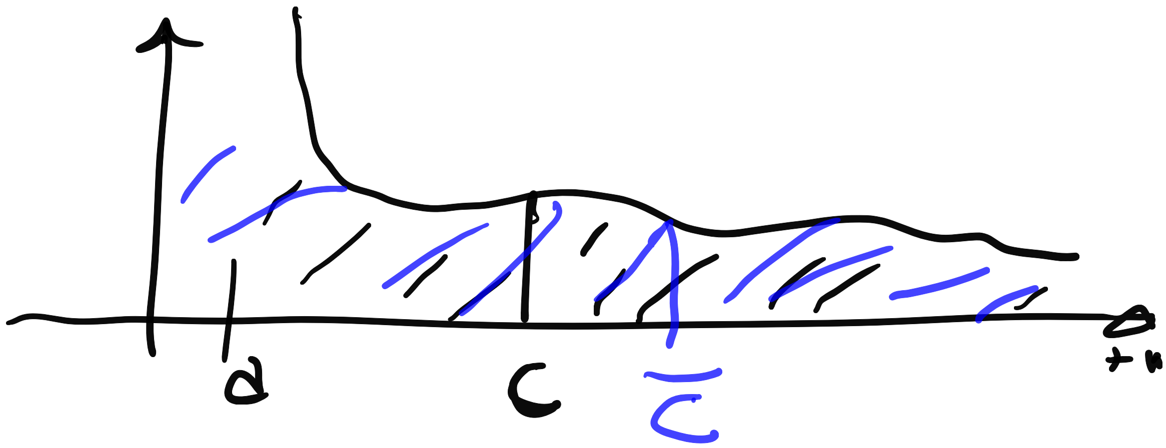
$$= \lim_{\bar a \to a^+} \left(\int_{\bar a}^{c} f(x)\,dx + \int_c^{\bar c} f(x)\,dx \right) = **$$

$$\boxed{\int_{\bar c}^{+\infty} f(x)\,dx} = \lim_{b \to +\infty} \left(\int_{\bar c}^{b} f(x)\,dx \right) =$$

$$= \lim_{b \to +\infty} \left(\int_{\bar c}^{c} f(x) + \int_c^{b} f(x)\,dx \right) = (***)$$

$$** = \boxed{\int_a^{c} f(x)\,dx} + \int_c^{\bar c} f(x)\,dx$$

$$*** = \boxed{\int_c^{b} f(x)\,dx} + \int_{\bar c}^{c} f(x)\,dx$$



$$f :]-\infty, -1] \cup [1, +\infty[\to \mathbb{R}$$

$$f(x) = \frac{1}{x^2}$$

$$\int_{]-\infty,-1]\cup[1,+\infty[} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{1}^{+\infty} f(x)\,dx$$

$$= \lim_{b\to-\infty} \int_{-b}^{-1} \frac{1}{x^2} + \lim_{d\to+\infty} \int_{1}^{d} \frac{1}{x^2}\,dx$$

$$= \lim_{b\to-\infty} -\frac{1}{x}\Big|_{-b}^{-1} + \lim_{d\to+\infty} -\frac{1}{x}\Big|_{1}^{d} =$$

$$\lim_{b\to+\infty}\left(1-\frac{1}{b}\right) + \left(-\frac{1}{d}+1\right) = \underline{2}$$

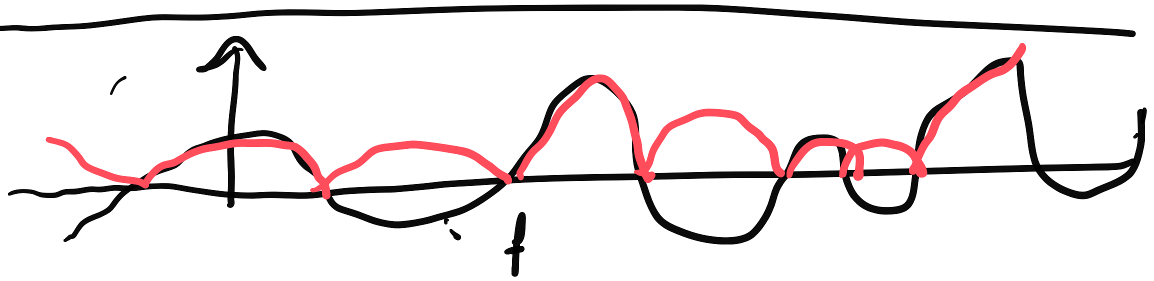
Definition: $f : I \longrightarrow \mathbb{R}$

(with $I =]a,b]$, $I = [a,b[$, $I =]-\infty,a]$, $I = [a,+\infty[$)

f is absolutely integrable in the generalized sense if $|f|$ is integrable in gen. sense

i.e.

$$\int_I |f(x)|\,dx < +\infty$$



Theorem 1

f is absolutely integrable in gen. sense $\Longrightarrow$ f is integrable in gen sense.

Idea of the proof:

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) > 0 \\ -f(x) & \text{if } f(x) \leq 0 \end{cases}$$

$$|f| = f^+ + f^-$$

$$f = f^+ - f^-$$

$$\boxed{f(x) = \frac{\sin x}{x}}$$

$f : [1, +\infty[\to \mathbb{R}$

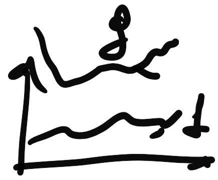
is integr. in gen sense but **not** absolutely integrable

The theorem "abs int $\Rightarrow$ int" is useful because we know several thing on generalized int of positive functions.

Theorem 2. Two functions

$f, g : [a, b[\to \mathbb{R}$ $(b \in \mathbb{R} \cup \{+\infty\})$

$$0 \le f(x) \le g(x) \quad \forall x \in [a, b[$$



f is integr. in gen. sense $\Longleftarrow$ g is integr. in gener. sense

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Example $h(x) = \dfrac{\sin x}{x^2}$ $\quad h : [1, +\infty[\to \mathbb{R}$

is h abs. int. ?

$$\int_1^{+\infty} |h(x)|\, dx = \int_1^{+\infty} \frac{|\sin x|}{x^2}\, dx$$

$$f(x) = \frac{|\sin x|}{x^2} \le \frac{1}{x^2} = g(x)$$

$$\int_1^{+\infty} \frac{1}{x^2}\, dx < +\infty \xRightarrow{th2} \int \frac{|\sin x|}{|x^2|}\, dx < +\infty$$

$$\xRightarrow{th. 1} \int \frac{\sin x}{x^2}\, dx \text{ is int.}$$

in gen. sense.

Proof: we have to prove

$$\lim_{\bar{b} \to b^-} \int_a^{\bar{b}} f(x)\, dx < +\infty$$

$$\varphi(\bar b) = \int_a^{\bar b} f(x)\,dx$$

Since $f(x) \ge 0$

$\varphi(\bar b)$ is increasing

so the limit of

$\lim_{\bar b \to b} \varphi(\bar b)$ exists equal to $\begin{cases} +\infty \\ \text{or} \\ r \ge 0 \end{cases}$

$$\varphi(\bar b) = \int_a^{\bar b} f(x)\,dx \le \int_a^{\bar b} g(x)\,dx$$

by assumption

$$\lim_{\bar b \to b} \int_a^{\bar b} g(x)\,dx < +\infty$$

$$\| $$

$$\int_a^b g(x)\,dx$$

$$\Rightarrow \varphi(\bar b) \le \boxed{\int_a^b g(x)\,dx}$$

$$\Longrightarrow \underset{\| }{\lim} \varphi(\bar b) \le \int_a^b g(x)\,dx.$$

$\int_a^b f(x)\,dx$ ~ exists

Example Study the integrability of

$$\int_1^{+\infty} \frac{\operatorname{arctg}(x)}{x^{\frac{3}{2}}}\,dx$$

$$f(x) = \frac{\operatorname{arctg}(x)}{x^{\frac{3}{2}}} \le \boxed{\frac{\frac{\pi}{2}}{x^{\frac{3}{2}}}} = g(x)$$

int. in gen sense because $\frac{3}{2} > 1$

yes, it is integrable.

Theorem $g, f : [a, b[\longrightarrow [0, +\infty[$ $f \geq 0$, $g \geq 0$

if $f = o(g)$ for $x \to b$

and g is integrable in gen. sense

$\Longrightarrow$ f is integrable in gen. sense

Theorem $f, g \geq 0$ ... ...

f is asymptotic to g for $x \to b$

f is integrab in gen. sense $\Longleftrightarrow$ g is integrable in gen. sense

Exercise: Study integrability

$$\int_1^{+\infty} \frac{\log x}{x^3}\, dx$$

Exercise: Studi absolute integrability when the parameter $\alpha \in \mathbb{R}$

$$\int_1^{+\infty} \left(\alpha \sin \frac{1}{x} - \alpha^2 \operatorname{arctg}\left(\log\left(1+\frac{1}{x}\right)\right)\right) dx$$