Series

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CONVERGENCE OF SERIES

FUNDAMENTAL SERIES

GENERALIZED ARMONIC SERIE

$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ $\alpha > 1$ converges

$\alpha \leq 1$ diverges

GEOMETRIC SERIE

$\sum_{n=0}^{+\infty} q^n$ $|q|<1$ converges at $\frac{1}{1-q}$

$q \geq 1$ diverges

$q \leq -1$ doesn't exist

TELESCOPING SERIE

$\sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges at 1

NECESSARY CONDITION

$\sum_{n=n_0}^{+\infty} a_n$ converges $\Longrightarrow$ $\lim_{n\to+\infty} a_n = 0$

limit of succession must be infinitesimal

$\sum_{n=n_0}^{+\infty} a_n$ converges $\Longrightarrow$ $\lim_{k\to+\infty} R_k = \lim_{k\to+\infty} \sum_{n=k}^{+\infty} a_n = 0$

limit of residuals must be infinitesimal

e.g. $\sum_{n=1}^{+\infty} 1+\frac{1}{n}$ since $\lim_{n\to+\infty} 1+\frac{1}{n} = 1 \neq 0$ the serie diverges

$\rightarrow$ we can use the test only to find out if the serie diverges !!

Criteria for alternating sign series $[a_n \gtrless 0]$

ABSOLUTE CONVERGENCE

$\sum_{n=n_0}^{+\infty} |a_n|$ converges (absolutely) $\Longrightarrow$ $\sum_{n=n_0}^{+\infty} a_n$ converges

NOTE: if $\sum_{n=n_0}^{+\infty} |a_n|$ diverges we cannot conclude anything about $\sum_{n=n_0}^{+\infty} a_n$

LEIBNIZ TEST

given $\sum_{n=n_0}^{+\infty} (-1)^n b_n$ with $b_n \geq 0$ $\forall n$

given $\sum_{n=n_0}^{+\infty} (-1)^n b_n$ with $b_n \geq 0 \ \forall n$

if $\begin{cases} \lim_{n\to+\infty} b_n = 0 \\ \{b_n\} \text{ is monotonic decreasing succession for } n \geq M \end{cases} \Longrightarrow \sum_{n=n_0}^{+\infty} (-1)^n b_n$ converges

e.g. $\sum_{n=n_0}^{+\infty} (-1)^n \frac{1}{n+1}$

- necessary condition: $\lim_{n\to+\infty} \left| (-1)^n \frac{1}{n+1} \right| = \lim_{n\to+\infty} \frac{1}{n} = 0$
- absolute convergence
  $\sum_{n=n_0}^{+\infty} \frac{1}{n+1} \underset{\text{comparison}}{\leq} \sum_{n=n_0}^{+\infty} \frac{1}{n}$ armonic series $\alpha = 1$ $\Rightarrow$ diverges

  ↳ we don't have absolute convergence but we can still have convergence
- $\begin{cases} \frac{1}{n+1} \geq 0 \ \forall n & ✓ \\ \lim_{n\to+\infty} \frac{1}{n+1} = \lim_{n\to+\infty} \frac{1}{n} = 0 & ✓ \\ \frac{1}{n+1} \geq \frac{1}{(n+1)+1} \quad n+2 \geq n+1 \ \forall n & ✓ \end{cases}$ the series converges for Leibniz

Criteria for positive-term series (PTS) $[a_n \geq 0 \ \forall n]$

COMPARISON TEST

if $a_n \leq b_n$ for $n \geq M$ $\begin{cases} \sum_{n=n_0}^{+\infty} b_n \text{ converges} \Longrightarrow \sum_{n=n_0}^{+\infty} a_n \text{ converges} \\ \sum_{n=n_0}^{+\infty} a_n \text{ diverges} \Longrightarrow \sum_{n=n_0}^{+\infty} b_n \text{ diverges} \end{cases}$

e.g. $\sum_{n=1}^{+\infty} \frac{1}{2n^2+n}$

- necessary condition $\lim_{n\to+\infty} \frac{1}{2n^2+n} \sim \frac{1}{2n^2} = 0$ ✓
- consider $\frac{1}{2n^2} \geq \frac{1}{2n^2+n}$ $n \geq 0$

  since $\frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges for armonic series $\alpha = 2$

  also our series converges

ASYMPTOTIC COMPARISON TEST

consider $\ell \triangleq \lim_{n\to+\infty} \frac{a_n}{b_n}$

$\begin{cases} \text{if } \ell \in]0;+\infty[& \sum_{n=n_0}^{+\infty} a_n \text{ converges} \Longleftrightarrow \sum_{n=n_0}^{+\infty} b_n \text{ converges} \\ \text{if } \ell = 0 & \sum_{n=n_0}^{+\infty} b_n \text{ converges} \Longrightarrow \sum_{n=n_0}^{+\infty} a_n \text{ converges} \\ \text{if } \ell = +\infty & \sum_{n=n_0}^{+\infty} b_n \text{ diverges} \Longrightarrow \sum_{n=n_0}^{+\infty} a_n \text{ diverges} \end{cases}$

$\left|\right.$ if $\ell = +\infty$ $\quad \sum_{n=n_0}^{+\infty} b_n$ diverges $\Rightarrow$ $\sum_{n=n_0}^{+\infty} a_n$ diverges

e.g. $\sum_{n=1}^{+\infty} \frac{1+e^{-n}}{\sqrt{n}}$

- n.c. $\lim_{n\to+\infty} \frac{1+e^{-n}}{\sqrt{n}} \sim n^{-1/2} = 0$ ✓
- consider $\sum_{n=1}^{+\infty} \frac{1}{n}$:

$$\begin{cases} \lim_{n\to+\infty} \left(\frac{1+e^{-n}}{\sqrt{n}}\right) \Big/ \left(\frac{1}{n}\right) = \lim_{n\to+\infty} \sqrt{n}(1+e^{-n}) \sim n^{1/2} = +\infty \\ \sum_{n=1}^{+\infty} \frac{1}{n} \text{ diverges (armonic serie } \alpha = 1) \end{cases}$$

$\rightarrow$ our serie diverges for asymtotic comparison test

RATIO TEST

consider $\ell \triangleq \lim_{n\to+\infty} \frac{a_{n+1}}{a_n}$ $\begin{cases} \text{if } \ell < 1 & \sum_{n=n_0}^{+\infty} a_n \text{ converges} \\ \text{if } \ell > 1 & \sum_{n=n_0}^{+\infty} a_n \text{ diverges} \end{cases}$

NOTE: if $\ell = 1$ we cannot conclude

e.g. $\sum_{n=3}^{+\infty} \frac{n+1}{(n-2)^3}$

- n.c. $\lim_{n\to+\infty} \frac{n+1}{(n-2)^3} \sim \frac{1}{n} = 0$ ✓
- consider the ratio test

$$\lim_{n\to+\infty} \left(\frac{(n+1)+1}{(n+1-2)^3}\right) \Big/ \left(\frac{n+1}{(n-2)^3}\right) = \lim_{n\to+\infty} \frac{n+2}{n+1}\left(\frac{n-2}{n-3}\right)^3$$

$$= \lim_{n\to+\infty} \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{n-3}\right)^3$$

$$\sim \lim_{n\to+\infty} \left(1+\frac{1}{n+1}\right)\left(1-\frac{3}{n-3} + o\left(\frac{1}{n}\right)\right)$$

$$\sim \lim_{n\to+\infty} 1 + \left(\frac{1}{n+1} - \frac{3}{n-3}\right) + o\left(\frac{1}{n}\right)$$

$$\sim \lim_{n\to+\infty} 1 + \frac{n-3-3n-3}{(n+1)(n-3)}$$

$$\sim \lim_{n\to+\infty} 1 - \frac{2n+6}{(n+1)(n-3)} = \boxed{1 = \ell}$$

thus we cannot conclude anything with the ratio test

- If we use the asymtotic comparison $\frac{n+1}{(n-2)^3} \sim \frac{1}{n^2}$ thus it converges

e.g. $\sum_{n=1}^{\infty} \frac{2n+1}{n!}$

- n.c. $\lim_{n\to+\infty} \frac{2n+1}{n!} = \lim_{n\to+\infty} \frac{2n+1}{n} \frac{1}{(n-1)!} \sim \lim_{n\to+\infty} \frac{1}{n!} = 0$ ✓
- consider $\frac{2(n+1)+1}{(n+1)!} \Big/ \frac{n!}{2n+1}$

$\lim \frac{2n+3}{\quad} \frac{n!}{\quad} = \lim \left(1+\frac{2}{\quad}\right) \frac{1}{n+1} = 0 < 1$

□ consider $\frac{\dots}{(n+1)!} \Big/ \frac{n!}{2n+1}$

$$\lim_{n\to+\infty} \frac{2n+3}{2n+1}\,\frac{n!}{(n+1)!} = \lim_{n\to+\infty}\left(1+\frac{2}{2n+1}\right)\frac{1}{n+1} = 0 < 1$$

thus the series converges for the ratio test

ROOT TEST

consider $\ell \triangleq \lim_{n\to+\infty} \sqrt[n]{a_n}$
$$\begin{cases} \text{if } \ell < 1 & \sum_{n=n_0}^{+\infty} a_n \text{ converges} \\ \text{if } \ell > 1 & \sum_{n=n_0}^{+\infty} a_n \text{ diverges} \end{cases}$$

NOTE: if $\ell = 1$ we cannot conclude

e.g. $\sum_{n=1}^{+\infty} n^{-n} - (n+1)^{-n}$

□ n.c. $\lim_{n\to+\infty} n^{-n} - (n+1)^{-n} = \lim_{n\to+\infty} e^{-n\log n} - e^{-n\log(n+1)} = 0$ ✓

□ consider $\lim_{n\to+\infty} \sqrt[n]{(n)^{-n} - (n+1)^{-n}} = \lim_{n\to+\infty}\left[\left(\frac{1}{n}\right)^n - \left(\frac{1}{n+1}\right)^n\right]^{1/n}$

$$= \lim_{n\to+\infty}\left(\frac{1}{n+1}\right)^{\frac{n}{n}}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{\frac{1}{n}} = \lim_{n\to+\infty}\frac{1}{n+1}\, e^{\frac{1}{n}\log\left(\underbrace{\left(1+\frac{1}{n}\right)^n}_{e} - 1\right)}$$

$$\sim \lim_{n\to+\infty}\frac{1}{n+1}\,\underbrace{e^{\frac{1}{n}\log(e-1)}}_{1} \sim \lim_{n\to+\infty}\frac{1}{n+1} = 0 < 1$$

thus our series converges for the root test

CAUCHY COMPRESSION TEST

$\sum_{n=n_0}^{+\infty} a_n$ converges $\Longleftrightarrow$ $\sum_{n=n_0}^{+\infty} 2^n a_{2^n}$ converges

e.g. $\sum_{n=2}^{+\infty} \frac{1}{n(\log n^2)^2}$

□ n.c. $\lim_{n\to+\infty} \frac{1}{n(\log n^2)^2} = 0$ ✓

□ consider $\sum_{n=n_0}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}(\log 2^{2n})^2} = \sum_{n=n_0}^{\infty} \frac{1}{4n^2(\log 2)^2}$

using the asymptotic comparison $\sim \frac{1}{n^2}$ with the armonic serie, the serie converges thus our original serie also converges

INTEGRAL TEST

consider $f(x)$ continuous function, $f(x) \geq 0$ and $f(x)$ decreasing

consider the series $\sum_{n=n_0}^{+\infty} a_n$ such that $a_n = f(n)$, $n \in \mathbb{N}$

$+\infty$ $+\infty$

consider the series $\sum_{n=n_0}^{+\infty} a_n$ such that $a_n = f(n)$, $n \in \mathbb{N}$

$$\sum_{n=n_0}^{+\infty} a_n \text{ converges} \iff \int_{n_0}^{+\infty} f(x)\,dx \text{ converges}$$

e.g. $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^{\alpha}(\log n)^{\beta}}$

$$\begin{cases} \text{for } \alpha > 1, \forall \beta & \text{converges} \\ \text{for } \alpha < 1, \forall \beta & \text{diverges} \\ \text{for } \alpha = 1, \beta > 1 & \text{converges} \\ \text{for } \alpha = 1, \beta \leq 1 & \text{diverges} \end{cases}$$

$\Downarrow$

It is another fundamental serie we can demonstrate using the integral test