MATH4U - 15/12/2022

Tuesday, 13 December 2022 11:16

1) QUESTIONS ABOUT INTEGRALS REVIEW

2) DEFINITE AND INDEFINITE INTEGRALS

3) GENERALIZED INTEGRALS

22) $\int \frac{3x}{x^3-1}\,dx$

$\frac{3x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} = \frac{Ax^2+Ax+A+Bx^2+Cx-Bx-C}{}$

$\frac{Ax+B}{x-1} + \frac{C}{x^2+x+1} = Ax^3 + \dots$

$\begin{cases} A+B=0 \\ A+C-B=3 \\ A-C=0 \end{cases}$ $\begin{matrix} B=-A \\ C=A \\ A+A-(-A)=3 \end{matrix}$ $\begin{matrix} A=1 \\ B=-1 \\ C=1 \end{matrix}$

$\int \frac{1}{x-1}\,dx + \int \frac{-x+1}{x^2+x+1}\,dx$

(A) (B)

(A) $= \log(x-1)+C$

(B) $= -\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx$

$-\frac{1}{2}\log(x^2+x+1)+C$

$x^2+2\left(\frac{1}{2}\right)x+\frac{1}{4}+\frac{3}{4}$

$\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$

$\left(\frac{\sqrt{3}}{2}\right)^2\left[\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right]$

$\frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]$

$\frac{3}{2}\int \frac{4\sqrt{3}}{3\cdot 2}\,\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx$

$\sqrt{3}\,\text{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

$\Rightarrow \log(x-1) - \frac{1}{2}\log(x^2+x+1) + \sqrt{3}\,\text{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$\int e^{2x}\,dx \quad e^x = y \quad \int y^2\,\frac{1}{y}\,dy$

28) $\int \frac{e^{2x}}{\sqrt{1+e^x}}dx$ $\quad e^x = y \longrightarrow \int \frac{y^2}{\sqrt{1+y}}\frac{1}{y}dy$

$x = \log y$

$d(x) = d(\log y)$

$dx = \frac{1}{y}dy$

$\sqrt{1+y} = t$

$y = t^2 - 1$

$dy = 2t\,dt$

$\int \frac{t^2-1}{t} \cdot 2t\,dt$

$\int 2t^2 - 1\,dt = \frac{2}{3}t^3 - t + C$

$= \frac{2}{3}(1+y)^{3/2} - (1+y)^{1/2} + C = \frac{2}{3}(1+e^x)^{3/2} - (1+e^x)^{1/2} + C$

$\frac{1}{\sqrt{1+x^2}}$

25) $\int \frac{1}{\sqrt{x}\sqrt{2x+1}}dx$ $\quad \sqrt{2x+1} = y$ $\quad \int \frac{1}{\frac{1}{\sqrt{2}}\sqrt{y^2-1}\,y} y\,dy$

$x = \frac{y^2-1}{2}$

$\sqrt{x} = \sqrt{\frac{y^2-1}{2}}$

$dx = y\,dy$

$\sqrt{2}\int \frac{1}{\sqrt{y^2-1}}dy$ $\quad y = \cosh t$

$dy = \sinh t\,dt$

$\int \frac{1}{\sqrt{ch^2t-1}} \sinh t\,dt = \int dt$

$t + C = \log(y + \sqrt{y^2-1}) + C = \log(\sqrt{2x+1} + \sqrt{2x}) + C$

$(y^2 = 2x+1)$

es) $\int \frac{1}{\sin x}dx$ $\quad \sin x = y$ $\quad \int \frac{1}{y\sqrt{1-y^2}}dy$

$x = \arcsin y$

$dx = \frac{1}{\sqrt{1-y^2}}dy$

$\longrightarrow \int \frac{\cos^2 x + \sin^2 x}{\sin x}dx = \int \left(\frac{\cos^2 x}{\sin x} + \sin x\right)dx$

$dx = \arctan 2t$

$\int \frac{\cos x + \sin x}{\sin x}\,dx = \int \left(\frac{\cos x}{\sin x} + \sin x\right) dx$

$t = \operatorname{tg}\frac{x}{2} \qquad \sin x = \frac{2t}{1+t^2} \qquad dx = \operatorname{arctg} 2t$

$x = \frac{t}{2} \qquad dx = \frac{1}{2}dt \qquad \int \frac{1}{2}\frac{1}{\sin 2t}\,dt$

$\int \frac{1}{2}\frac{1}{\sin t \cos t}\,dt = \int \frac{1}{2}\frac{1}{\frac{\sin t}{\cos t}}\frac{1}{\cos^2 t}\,dt$ ($\frac{\sin t}{\cos t} \to \operatorname{tg} t$, $\frac{1}{\cos^2 t} \to \operatorname{tg}' t$)

$= \frac{1}{2}\log \operatorname{tg} t + c = \frac{1}{2}\log \operatorname{tg}\frac{x}{2} + c$

$\int \frac{1+t^2}{2t}\,\frac{1}{1+(2t)^2}\cdot\frac{1}{2}\,dt$

SOLUTIONS WITH OTHER METHODS

(Method 1) Integral of 1/sin(x) (substitution + trig identities + substitution)

https://www.youmath.it/forum/analisi-1/25459-integrale-indefinito-di-1sinx.html

ES) $\int_{-\pi}^{\pi} \frac{|\sin x|}{4-\cos^2 x}\,dx = \int_{-\pi}^{0} \frac{-\sin x}{4-\cos^2 x}\,dx + \int_{0}^{\pi} \frac{\sin x}{4-\cos^2 x}\,dx$

odd function (without 1·1) $\int_{-a}^{a} f(x)\,dx = 0$

$= 2\int_{0}^{\pi} \frac{\sin x}{4-\cos^2 x}\,dx$

$\cos x = t$

$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-t^2}$

$x = \arccos t$

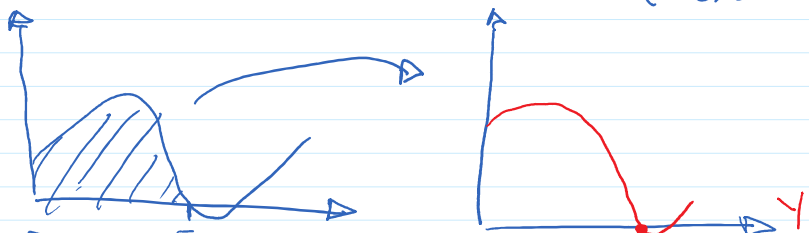
$dx = \frac{1}{-\sqrt{1-t^2}}\,dt$

$\cos 0 = 1$

$\cos \pi = -1$

$= 2\int_{1}^{-1} \frac{\sqrt{1-t^2}}{4-t^2}\,\frac{1}{-\sqrt{1-t^2}}\,dt = 2\int_{-1}^{1} \frac{1}{4-t^2}\,dt = \int_{-1}^{1} \frac{-2}{t^2-4}\,dt$

$(t-2)(t+4)$



$x = 2y \qquad \pi = 2y^* \to y^* = \frac{\pi}{2}$

$$\frac{A}{t-2} + \frac{B}{t+2} = \frac{(A+B)t + 2A - 2B}{}$$

$$\begin{cases} A+B=0 \\ 2A-2B=-2 \end{cases}$$

$$\begin{cases} B=-A \\ 4A=-2 \end{cases} \qquad \begin{matrix} A=1/2 \\ B=-1/2 \end{matrix}$$

$$\int \frac{1}{2}\frac{1}{t-2}\,dt - \int \frac{1}{2}\frac{1}{t+2}\,dt = \frac{1}{2}\log|t-2| - \frac{1}{2}\log|t+2| + C$$

$$= \frac{1}{2}\log\left|\frac{t-2}{t+2}\right| + C$$

$$\int_{-1}^{1} dt = \int_0^{\pi} dx = \frac{1}{2}\log\left|\frac{t-2}{t+2}\right|_{t=-1}^{t=1} = \left(\frac{1}{2}\log\left|\frac{\cos x - 2}{\cos x + 2}\right|\Big|_{x=0}^{x=\pi}\right)$$

$$= \frac{1}{2}\log\left|-\frac{1}{3}\right| - \frac{1}{2}\log\left|\frac{-3}{1}\right| = -\frac{1}{2}\log 3 - \frac{1}{2}\log 3 = \boxed{-\log 3}$$

$$\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}$$

1) $\int_0^1 \frac{\cos x}{\log x}\,dx$ domain: $x>0$, $\log x \neq 0$ $\qquad x>0 \wedge x\neq 1$

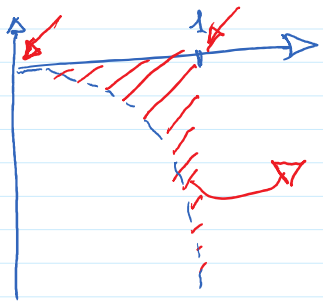
$\lim_{a\to 1} \int_0^a \frac{\cos x}{\log x}\,dx$

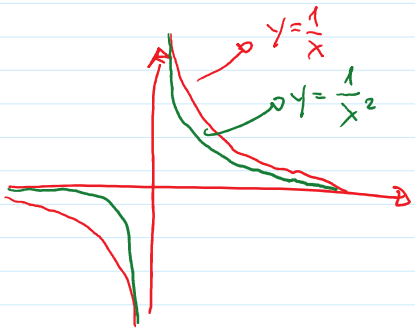
$\lim_{a\to 0} \int_a^1 \frac{\cos x}{\log x}\,dx \Longrightarrow$ $\frac{\cos x}{\log x}$ is extendible with continuity in 0

$\Downarrow$

$\int dx$ converges

$\int_a^b f(x)\,dx$ converges iff $\int_a^b f(x)\,dx = \ell \begin{matrix} < +\infty \\ > -\infty \end{matrix}$





$$\lim_{x\to 1^-} \frac{\cos x}{\log x} \overset{x=1-y}{=} \lim_{y\to 0^+} \frac{\cos(1-y)}{\log(1-y)} \sim \lim_{y\to 0^+} \frac{\alpha}{-y+o(y)} \sim \frac{1}{y}$$

$$\int_0^{x_0} \frac{1}{x^\beta}\,dx \quad \text{converges iif } \beta > 1$$

INTEGRAL DIVERGES

EXERCISES ABOUT INTEGRALS

21) $\int 2x \log(x-5)\,dx$

22) $\int \frac{3x}{x^3-1}\,dx$

23) $\int \frac{x^5-x+1}{x^4+x^2}\,dx$

24) $\int \frac{1}{(e^x+1)^2}\,dx$

25) $\int \frac{1}{\sqrt{x}\sqrt{2x+1}}\,dx$

26) $\int \frac{1}{\sqrt{x}(2x+1)}\,dx$

27) $\int_{-2}^{0} x \sin x^2 e^{2x^2}\,dx$

28) $\int_{\pi/2}^{} x\sqrt{4x-1}\,dx$

29) $\int_0^{} \sqrt{1+\sin^2 x}\,\sin 2x\,dx$

30) $\int_0^{\pi/3} \frac{\operatorname{tg} x}{1+\log\cos x}\,dx$

31) $\int x(e^{x^2}+2e^{-3x})\,dx$

32) $\int \frac{\sin 2x \sin x}{2+\sin x}\,dx$

33) $\int_1^e x^2 \log x\,dx$

34) $\int x \operatorname{arctg} x\,dx$

35) $\int x\cos^2 x\,dx$

36) $\int \frac{x^2+1}{x^2+2x-3}\,dx$

37) $\int \frac{1+x}{x^2+4x+6}\,dx$

38) $\int e^{3x}\sin x\cos x\,dx$

39) $\int \frac{\sqrt{3+x^2}}{x}\,dx$

40) $\int_0^2 x\sqrt{|3-x^2|}\,dx$