$$\int \varphi(f(x)) \cdot f'(x)\,dx =$$

$$y = f(x)$$
$$dy = f'(x)\,dx$$
$$dx = \frac{dy}{f'(x)}$$

$$\int \varphi(y) \cdot \frac{\cancel{f'(x)}\,dy}{\cancel{f'(x)}} =$$

$$\int \varphi(y)\,dy$$

Example

$$\int \frac{1}{(x^2+1)[\operatorname{arctg}^2 x + 2\operatorname{arctg} x + 3]}\,dx = :$$

$$\varphi(y) = \frac{1}{y^2+2y+3}$$

$$y = f(x) = \operatorname{arctg}(x)$$

$$\left(= \int P(f(x))\, f'(x) \right) =$$

$$y = \operatorname{arctg} x$$

$$dy = \frac{dx}{1+x^2}$$

$$\int \frac{1}{\underbrace{y^2+2y+3}_{\shortparallel}}\, dy$$

$$\Delta = \quad 4-12<0$$

$$\int \frac{1}{(y^2+2y+1)+2}\, dy = \int \frac{dy}{(y+1)^2+2} =$$

$$\frac{1}{2}\int \frac{dy}{\left(\frac{y+1}{\sqrt{2}}\right)^2+1} =$$

$$z = \frac{y+1}{\sqrt{2}}$$

$$dz = \frac{dy}{\sqrt{2}}$$

$$dy = \sqrt{2}\, dz$$

$$= \frac{1}{2}\int \frac{\sqrt{2}\, dz}{(z^2+1)} =$$

$$\shortparallel$$

$$\frac{\sqrt{2}}{2} \operatorname{arctg} z = \frac{\sqrt{2}}{2} \operatorname{arctg}\left(\frac{y+1}{\sqrt{2}}\right) =$$

$$= \frac{\sqrt{2}}{2} \operatorname{arctg}\left(\frac{\operatorname{arctg} x + 1}{\sqrt{2}}\right)$$

Example

$$\int \frac{\cosh(x)\sinh(x)}{\cosh^2(x) + 2\cosh x + 1}\,dx!$$

$$= \int \frac{y}{y^2 + 2y + 1}\,dy =$$

$y = \cosh(x)$

$dy = \sinh(x)\,dx$

$$\int \frac{y}{(y+1)^2}\,dy = A\log|y+1| + \frac{B}{(y+1)}$$

$$\frac{y}{(y+1)^2} = \frac{A}{(y+1)} + \frac{B}{(y+1)^2}$$

(determine A and B)

Example

$$\int \frac{\cosh x \; dx}{\cosh^2 x + 2\cosh x + 1} =$$

$$\frac{1}{2}\int \frac{(e^x + e^{-x})\,dx}{\frac{e^{2x} + e^{-2x} + 2}{4} + 2\left(\frac{e^x + e^{-x}}{2}\right) + 1} :$$

$$= \frac{1}{2}\int \frac{e^{2x}(e^x + e^{-x})\,dx}{e^{2x}\left[\left(\frac{e^{2x} + e^{-2x} + 2}{4}\right) + e^x + e^{-x} + 1\right]}$$

$$= \frac{1}{2} \int \frac{e^{3x} + e^{x} \, dx}{\frac{e^{4x}}{4} + \frac{1}{4} + \frac{2e^{2x}}{4} + e^{3x} + e^{x} + e^{2x}}$$

$$= \frac{1}{2} \int \frac{e^{x}\left(e^{2x}+1\right) dx}{\frac{e^{4x}}{4} + e^{3x} + \frac{3}{2}e^{2x} + e^{x} + \frac{1}{4}} =$$

$$\underset{\substack{y = e^{x} \\ dy = e^{x} dx}}{=} \frac{1}{2} \int \frac{y \, dy}{\frac{y^4}{4} + y^3 + \frac{3}{2}y^2 + y + \frac{1}{4}} =$$

$$= \frac{1}{8} \int \frac{y \, dy}{y^4 + 4y^3 + 6y^2 + 4y + 1} =$$

$$= \frac{1}{8} \int \frac{y \, dy}{(1+y)^4} =$$

$$= \ldots\ldots$$

Example

$$\int_4^6 \frac{\sqrt{x}+5}{x+2\sqrt{x}}\,dx \underset{1}{=} 2\int_2^{\sqrt{6}} \frac{(y+5)y}{y^2+2y}.$$

$y=\sqrt{x}$

$dy=\frac{1}{2\sqrt{x}}dx$

$dx = dy\,2\sqrt{x} =$

$= 2dy\,y$

$$= 2\int_2^{\sqrt{6}} \frac{y^2+5y}{y^2+2y}\,dy = 2\int_2^{\sqrt{6}} \frac{y^2+2y+3y}{y^2+2y} =$$

$$= 2\int_2^{\sqrt{6}} \left(1+\frac{3y}{y^2+2y}\right)dy = \dots$$

Example

$$\int \frac{\sqrt[3]{x}-5\sqrt{x}}{\sqrt[6]{x}+1}\,dx \underset{1}{=} 6\int \frac{(y^2-5y^3)}{y+1}y^5 =$$

$y=\sqrt[6]{x}$

$dy = \frac{1}{6}x^{-\frac{5}{6}} =$

$= \frac{1}{6}\frac{1}{(\sqrt[6]{x})^5}dx$

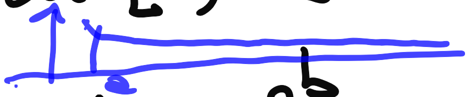
$dx = 6dy\,(\sqrt[6]{x})^5 =$

$= 6y^5 dy$

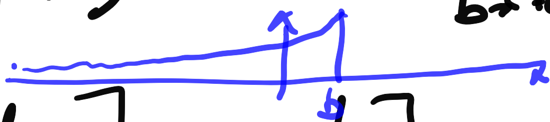
Geralized integrals.

Definition: $f : [a, +\infty[\longrightarrow \mathbb{R}$

- s.t. f is integrable on $[a, b]$ for every $b \geq a$.



$$\int_a^{+\infty} f(x)\,dx := \lim_{b \to +\infty} \int_a^b f(x)\,dx$$



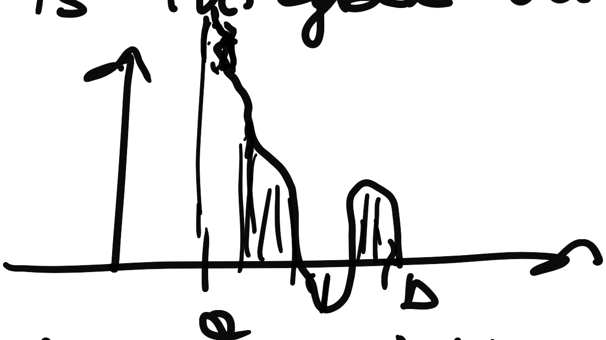
- Also $f :]-\infty, b] \longrightarrow \mathbb{R}$

s.t. f is integrable in $[a, b]$ $\forall a \leq b$

$$\int_{-\infty}^b f(x)\,dx := \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

- $f :]a, b] \longrightarrow \mathbb{R}$

f is integrable on every



s.t. f is integrable on every interval $]\bar{a}, b]$ $\forall \bar{a} \in]a, b]$

f is integrable in generalized sense if there exists

$$\int_a^b f(x)\,dx := \lim_{\bar{a}\to a^+} \int_{\bar{a}}^b f(x)\,dx$$

$f : [a, b[$ . . . . .

. . . . .

$$\int_a^b f(x)\,dx := \lim_{\bar{b}\to b^-} \int_a^{\bar{b}} f(x)\,dx \; .$$

We have seen that

$$\int_1^{+\infty} \frac{1}{x} \qquad \text{not integrable}$$

because $\displaystyle \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx = +\infty$

$$\int_1^{+\infty} \frac{1}{x^{\frac{1001}{1000}}}\,dx \text{ "}< +\infty\text{"}$$

short way to mean integrable in generalized sense.

Fact: if $\alpha > 1$

$\displaystyle \int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ exists, i.e. $\frac{1}{x^\alpha}$ is int. in gen. sense

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx := \lim_{b\to+\infty} \int_1^b \frac{1}{x^\alpha} dx =$$

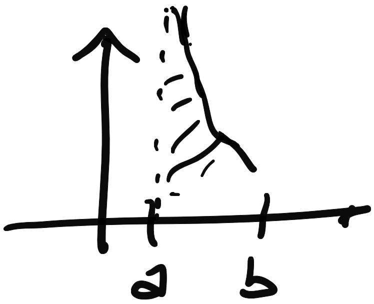
$$\lim_{b\to+\infty} \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_1^b = \lim_{b\to+\infty}\left(\frac{b^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) =$$

$$= -\frac{1}{-\alpha+1} = \frac{1}{\alpha-1}.$$

Show that $\displaystyle\int_\eta^{+\infty} \frac{1}{(x+d)^\alpha} dx$

$(\eta > -d)$ is int. in gen. sense
$\alpha > 1$

$$\int_a^b \frac{1}{(x-a)^\alpha} dx$$



$$\lim_{\bar a\to a+} \int_{\bar a}^b \frac{1}{(x-a)^\alpha} = \begin{cases} \frac{(x-a)^{1-\alpha}}{1-\alpha}\Big|_{\bar a}^b & \text{if } \alpha\neq 1 \\ \log|x-a|\,\Big|_{\bar a}^b & \alpha=1 \end{cases}$$

for $\boxed{\alpha < 1}$

$$\lim_{\bar a\to a+} -\frac{(\bar a-a)^{1-\alpha}}{1-\alpha} + \frac{(b-a)^{1-\alpha}}{1-\alpha} = \frac{(b-a)^{1-\alpha}}{1-\alpha}$$

$\alpha = 1$

$$= \lim_{\bar{a} \to a} \log |x-a| \Big|_{\bar{a}}^{b} :$$

$$\lim \left(\log |b-a| - \underbrace{\log |\bar{a}-a|}_{\downarrow\, -\infty} \right)$$

$\alpha > 1$ . . . . . . $\int = +\infty$.

not integrable

$\dfrac{1}{x^{\alpha}}$

is integr. in gen. sense on $[a, +\infty[$ if and only if $\alpha > 1$

. . . . on $]0, b]$ if and only if $\alpha < 1$

What should be the definition of int. in gen sense for $\alpha > 0$.

$$\int_{0}^{+\infty} \frac{1}{x^{\alpha}}\, dx$$

One idea might be

$$\int_0^{+\infty} \frac{1}{x^\alpha}\,dx = \lim_{n\to+\infty} \int_{\frac{1}{n}}^{n}$$

<u>Wrong</u>

The right def. is

<u>Def</u> $$\int_0^{+\infty} \frac{1}{x^\alpha}\,dx =$$

$c \in \,]0, +\infty[$,

if $\displaystyle\int_c^{+\infty} \frac{1}{x^\alpha}\,dx$ exists

and if $\displaystyle\int_0^{c} \frac{1}{x^\alpha}\,dx$ exists

then we set

$$\int_0^{+\infty} f(x)\,dx := \int_0^{c} \frac{1}{x^\alpha}\,dx + \int_c^{+\infty} \frac{dx}{x^\alpha}$$