$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{2x}{2x^2-5x+2}\,dx \qquad \boxed{\Delta = 25-16 = 9 > 0}$$

$$x_1, x_2 = \frac{5 \pm \sqrt{9}}{4} \quad \begin{matrix} 2 \\ \frac{1}{2} \end{matrix}$$

$$\frac{2x}{2x^2-5x+2} = \frac{A}{2(x-2)} + \frac{B}{\left(x-\frac{1}{2}\right)}$$

$$2x = A\left(x-\frac{1}{2}\right) + 2B(x-2)$$

$$2x = x(A+2B) + \left(-\frac{A}{2} - 4B\right)$$

$$\begin{cases} 2 = A+2B \\ 0 = -\frac{A}{2} - 4B \end{cases} \qquad \begin{cases} 4 = 2A+4B \\ 0 = -\frac{A}{2} - 4B \end{cases}$$

$$\begin{cases} 4 = \frac{3}{2}A \\ \end{cases} \qquad \begin{cases} A = \frac{8}{3} \\ 0 = -\frac{\frac{8}{3}}{2} - 4B \end{cases}$$

$$\begin{cases} A = \frac{8}{3} \\ 4B = -\frac{8}{6} \end{cases} \qquad \begin{cases} A = \frac{8}{3} \\ B = -\frac{2}{6} = -\frac{1}{3} \end{cases}$$

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \dots = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{\frac{8}{3}}{2(x-2)} - \frac{\frac{1}{3}}{\left(x-\frac{1}{2}\right)} =$$

$$\frac{4}{3}\log|x-2| - \frac{1}{2}\log\left(\left|x-\frac{1}{2}\right|\right)\Bigg|_{\frac{1}{2}}^{\frac{3}{2}} =$$

$$\left[\log|x-2|^{\frac{4}{3}} - \log\left(x-\frac{1}{2}\right)^{\frac{1}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \log\frac{|x-2|^{\frac{4}{3}}}{\left|x-\frac{1}{2}\right|^{\frac{1}{2}}}\Bigg|_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\boxed{\log} = \log_e$$

ln

$$\int \frac{a_0+a_1x}{ax^2+bx+c}\,dx \qquad \boxed{\Delta = b^2-4ac=0}$$

$$x_1=x_2= = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-b}{2a}$$

$$\int \frac{a_0+a_1x}{a\left(x+\frac{b}{2a}\right)^2}\,dx =$$

$$\frac{1}{a}\int \frac{A}{\left(x+\frac{b}{2a}\right)} + \frac{B}{\left(x+\frac{b}{2a}\right)^2}$$

$$= \frac{a_0 + a_1 x}{a(\cdots\cdots)} = \frac{1}{a}\,\frac{A\left(x+\frac{b}{2a}\right) + B}{(\cdots\cdots\cdots)}$$

$$\Updownarrow$$

$$\begin{cases} a_0 = \frac{Ab}{2a} + B \\ a_1 = A \end{cases}$$

$$\int = \frac{1}{a}\left(A \log\left|x+\frac{b}{2a}\right| - \frac{B}{\left(x+\frac{b}{2a}\right)}\right)$$

$$\int \frac{a_0 + a_1 x}{a x^2 + b x + c} \qquad \boxed{\Delta = b^2 - 4ac < 0}$$

model case:

$$\int \frac{a_0 + a_1 x}{\boxed{1 + x^2}} = \int \frac{a_0}{1+x^2} + a_1 \frac{2x}{2(1+x^2)}.$$

$$= a_0 \operatorname{arctg}(x) + \frac{a_1}{2} \log(1+x^2)$$

Let us study

$$\int \frac{x+1}{\boxed{x^2+x+4}} = \qquad \underbrace{\Delta = 1 - 16 < 0}.$$

$$\int \frac{x+1}{\underbrace{x^2+x+\frac{1}{4}}_{(x+\frac{1}{2})^2} + \left(4 - \frac{1}{4}\right)} dx = \int \frac{(x+1)\,dx}{(x+\frac{1}{2})^2 + \frac{15}{4}}.$$

$$= \frac{1}{\frac{15}{4}} \int \frac{(x+1)\,dx}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{15}{4}}}\right)^2 + 1} = \!=$$

$$y(x) = y = 2\frac{x+\frac{1}{2}}{\sqrt{15}}$$

$$2\left(x+\frac{1}{2}\right)=\sqrt{15}\,y \qquad \frac{dx}{dy}=\frac{\sqrt{15}}{2}$$

$$x+\frac{1}{2}=\frac{\sqrt{15}}{2}y$$

$$x=\frac{\sqrt{15}}{2}y-\frac{1}{2}$$

$$=\frac{4}{15}\frac{\sqrt{15}}{2}\int\frac{\left(\boxed{\frac{\sqrt{15}}{2}}y-\frac{1}{2}+1\right)}{(y^2+1)}dy$$

$$=\frac{1}{2}\frac{\cancel{2}}{\sqrt{15}}\cancel{\frac{\sqrt{15}}{2}}\int\frac{2y}{y^2+1}dy+$$

$$+\cancel{\frac{1}{2}}\frac{\cancel{2}}{\sqrt{15}}\int\frac{1}{y^2+1}dy=$$

$$=\frac{1}{2}\log(y^2+1)+\frac{1}{\sqrt{15}}\operatorname{arctg}y=$$

$$=\frac{1}{2}\log\left(y^2(x)+1\right)+\frac{1}{\sqrt{15}}\operatorname{arctg}y(x)$$

In general

$$\int \frac{a_0 + a_1 x}{ax^2 + bx + c} \quad \text{with } \Delta = b^2 - 4c < 0$$

$$\int \frac{1}{x^3 - 3x^2 + x - 3} \, dx =$$

$$\int \frac{1}{(x^2+1)(x-3)} \, dx = \int \frac{Ax+B}{x^2+1} + \frac{C}{x-3} =$$

$$1 = (x-3)(Ax+B) + C(x^2+1)$$

$$1 = -3B + C$$

$$0 = B - 3A + C$$

$$0 = A + C$$

$$\int = B \arctg(x) + \frac{A}{2} \log(1+x^2) + C \log|x-3|$$

Generalized integrals

Up to now:

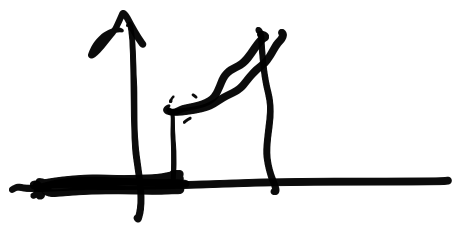
$$\int_a^b f(x)\,dx$$

- the integral is computed over a bounded and closed interval $[a,b]$
- the function f is bounded

$$\mathcal{R}([a,b])$$ ← set of Riemann integrable functions on $[a,b]$

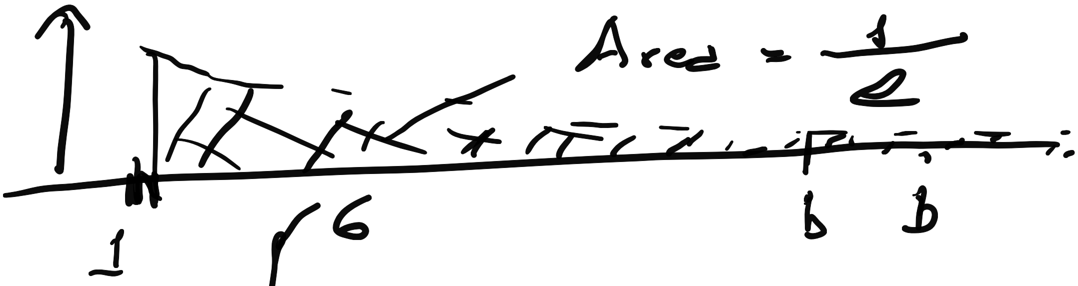
$$\cap$$

$$\mathcal{C}([a,b])$$

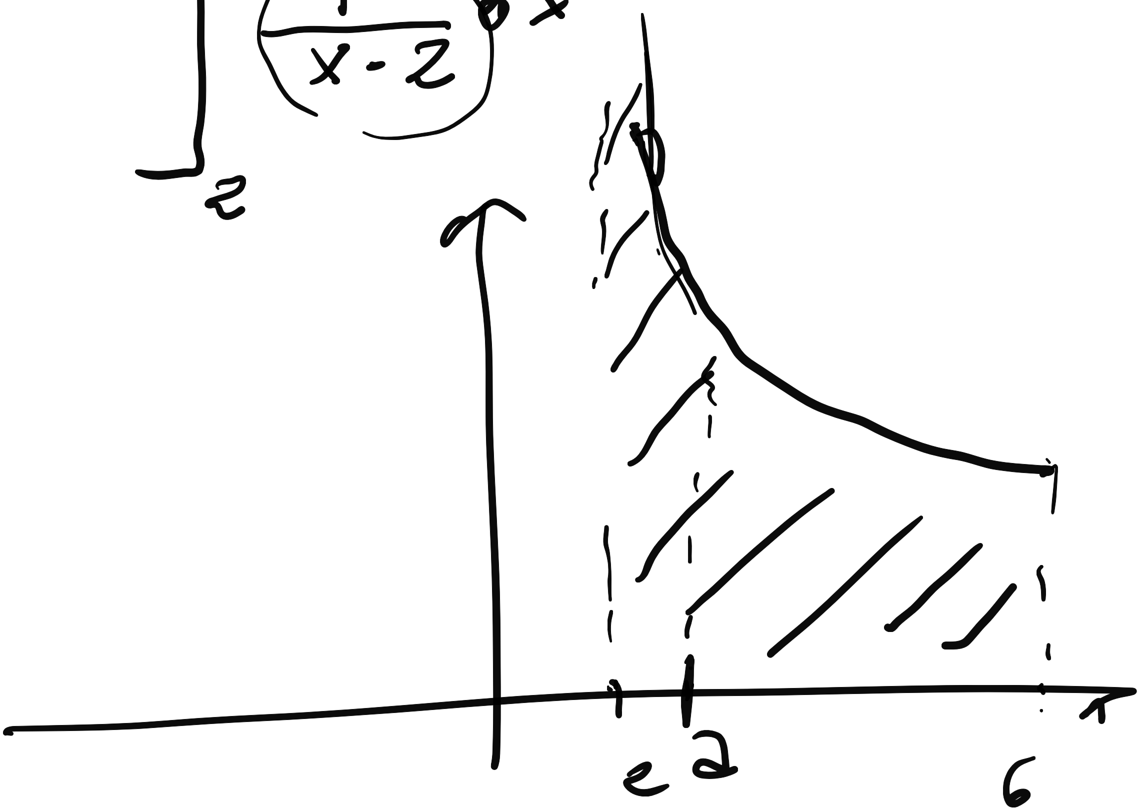


Can we give a reasonable meaning to

$$\int_1^{+\infty} e^{-x}\,dx \; ?$$



$$\int_2^6 \frac{1}{x-2}\,dx$$



Propose:

$$\int_1^{+\infty} e^{-x}\,dx = \lim_{b \to +\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b \to +\infty} \left(-e^{-x} \Big|_1^b \right) = \lim_{b \to +\infty} \left(-\underbrace{e^{-b}}_{0} + e^{-1} \right) =$$

$$= e^{-1} = \frac{1}{e}$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{b\to+\infty}\left(\int_1^b \frac{1}{x}\,dx\right) =$$

$$\lim_{b\to+\infty}\left(\log x \Big|_1^b\right) =$$

$$\lim_{b\to+\infty}\left(\log b\right) = +\infty$$

$$\int_1^{+\infty} \frac{1}{x^{\frac{1001}{1000}}} = \lim_{b\to+\infty}\left(\frac{x^{\left(1-\frac{1001}{1000}\right)}}{\left(1-\frac{1001}{1000}\right)}\right)\Bigg|_1^b =$$

$$= \lim_{b\to+\infty}\left(\frac{b^{-\frac{1}{1000}}}{-\frac{1}{1000}} - \frac{1}{-\frac{1}{1000}}\right) =$$

$$= 1000$$