$f : [a,b] \to \mathbb{R}$ bounded

$\int_a^b f(x)\,dx$ definition with subdivisions

integral (Riemann) [Lebesgue]

- How to compute $\int_a^b f(x)\,dx$?
- Find any primitive off $F : [a,b] \to \mathbb{R}$ of f, $(F' = f)$ then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

provided f is continuous.

So the area problem has been transformed into the primitive problem.

- For some f the primitive is immediate:

For instance:

$f(x) = x^{\alpha}$ $\qquad F(x) = \dfrac{x^{\alpha+1}}{\alpha+1}$ $(\alpha < -1)$

$f(x) = \underbrace{\dfrac{1}{x}}$, $\qquad F(x) = \log|x|$

$f(x) = \dfrac{1}{1+x^2}$ $x \in \mathbb{R}$ $\qquad F(x) = \operatorname{arctg}(x) + C$

$f(x) = \dfrac{1}{\sqrt{1-x^2}}$ $x \in]-1,1[$ $\qquad F(x) = \arcsin(x)$

$f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$ " $\qquad F(x) = \arccos(x)$

$f(x) = \sin x$

$f(x) = \cos x$

$\vdots$

$f(x) = \overline{2x \dfrac{\sin(x^2)}{2}} = -\dfrac{1}{2}\cos(x^2)$

Integration methods:

- **By parts:**

$$\int_a^b h'(x)\cdot k(x) = \boxed{h\cdot k\Big|_a^b - \int_a^b h\cdot k'(x)\,dx}$$

$$= h(b)k(b) - h(a)k(a) - \int_a^b h(x)\,k'(x)\,dx$$

- **Substitution:**

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\gamma(y))\frac{d\gamma}{dy}\,dy$$

where

$$\gamma : [\alpha, \beta] \longrightarrow [a, b]$$
$$y \longmapsto \gamma(y)$$

is of class C^1.

$$\int_0^1 \underbrace{x}_{h'}\, \underbrace{\operatorname{arctg}(x)}_{k}\, dx = \frac{x^2}{2}\operatorname{arctg} x \Big|_0^1$$

$$- \int_0^1 \frac{x^2}{2}\, \frac{1}{1+x^2}\, dx =$$

$$\frac{1}{2}\,\frac{\pi}{4} - \frac{1}{2}\int_0^1 \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \right) dx =$$

$$= \frac{1}{2}\,\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\operatorname{arctg} x \Big|_0^1 =$$

$$= \frac{1}{2}\,\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\,\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2}.$$

$$\int_{-1}^{1} (\arcsin x)\,dx = \int_{-1}^{1} 1\cdot \arcsin x \,dx =$$

$$x \arcsin x \Big|_{-1}^{1} - \int_{-1}^{1} x \frac{1}{\sqrt{1-x^2}}\,dx =$$

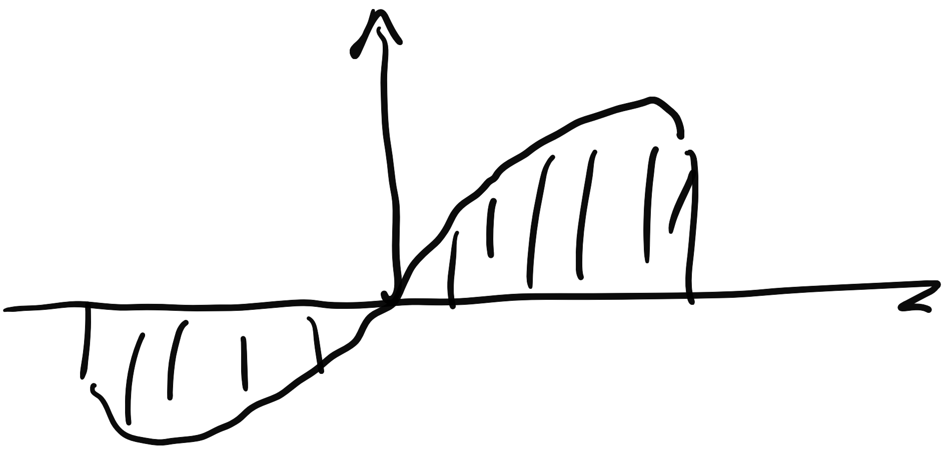
$$1\frac{\pi}{2} - \left(-1\cdot\left(-\frac{\pi}{2}\right)\right)$$

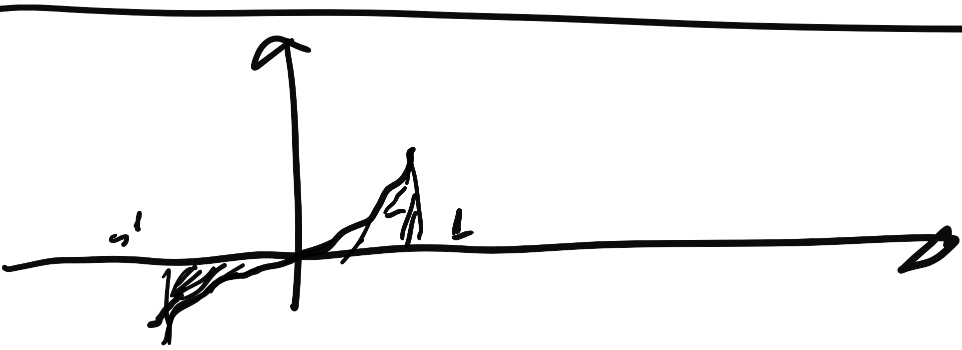
$$\underbrace{\frac{\pi}{2} - \frac{\pi}{2}} - \left(-\int_{-1}^{1} \frac{-x}{\sqrt{1-x^2}}\right) =$$

$$\frac{-x}{\sqrt{1-x^2}} = \left(\sqrt{1-x^2}\right)' = \frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}}$$

$$\pi + \left(\sqrt{1+x^2}\right)\Big|_{-1}^{1} =$$

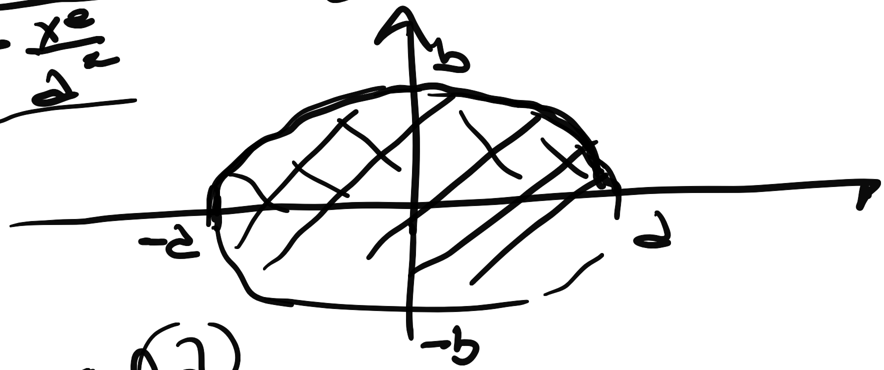
$$= \underbrace{\pi} + \left(\sqrt{2} - \sqrt{2}\right) = \pi.$$





Compute the area of the ellipse $\left\{(x,y) : \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}\right\}$

$$y = b\sqrt{1-\frac{x^2}{a^2}}$$



$$\text{Area} = 2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx =$$

$x = a\cos t$

$t \in [0,\pi]$

$\sqrt{1-\frac{a^2\cos^2 t}{a^2}} =$

$$= 2b\int_{\pi}^{0}\sin t\,(a\cos t)'\,dt$$

$$= -2ba\int_{\pi}^{0}\sin^2 t\,dt =$$

$$= 2ba\int_{0}^{\pi}\sin^2 t\,dt = 2ab\frac{\pi}{2}$$

$$\boxed{\int_0^{\pi}\sin^2 t} = -\sin t\cos t\Big|_0^{\pi} - \int_0^{\pi}(-\cos t)(\cos t) =$$

$$= +\int_0^{\pi}\cos^2 t =$$

$$= \boxed{\int_0^{\pi} (1 - \sin^2 t)\, dt}$$

$$2 \int_0^{\pi} \sin^2 dt = \int_0^{\pi} 1\, dt = \pi$$

$$\int_0^{\pi} \sin^2 dt = \boxed{\frac{\pi}{2}}$$

RATIONAL FUNCTIONS

$$\boxed{f(x) = \frac{P(x)}{Q(x)}}$$ with P and Q polinomials

$$\int \frac{x^3 + 2x^2 + 1}{x^2 + 1}\, dx$$

$$\int \frac{P(x)}{Q(x)}\, dx$$

we can <del>~~okay~~</del> : : assume degree(P) < degree(Q).

$$\frac{P(x)}{Q(x)} = \boxed{Z(x)} + \boxed{\frac{R(x)}{Q(x)}}$$

with degree of $R <$
$<$ degree of Q.

$$\begin{array}{l|l} x^3+2x^2+1 & x^2+1 \\ \hline x^3 \quad\quad +x & x+2 \\ \hline 2x^2-x+1 & \\ 2x^2 \quad\quad +2 & \\ \hline \boxed{-x-1} & \end{array}$$

$$\int \frac{x^3+2x^2+1}{x^2+1} = \int \left((x+2) - \frac{x+1}{x^2+1} \right) dx$$

$$= \frac{x^2}{2} + 2x - \int \frac{x+1}{x^2+1} dx =$$

$$= \frac{x^2}{2} + 2x - \frac{1}{2}\int \frac{2x}{x^2+1}$$

$$- \int \frac{1}{x^2+1} =$$

$$= \frac{x^2}{2} + 2x - \operatorname{arctg}(x) - \frac{1}{2} \log(x^2+1)$$

$$\frac{d}{dx}\left(\log(x^2+1)\right)$$

$$\frac{1}{x^2+1}\cdot(2x)$$

$$\Big(f(x^2+1)\Big)' = f'(x^2+1)\cdot 2x$$

$$\frac{1}{x^2+1}\cdot 2x$$

When we write $\log x$ we mean $\log_e x$

(i.e. the basis is $\underline{e}$, $\underline{\text{not}}$ 10)

$$\frac{P(x)}{Q(x)} \quad \text{with } \deg(P) < \deg(Q)$$

$n = \text{degree}(Q)$

$n = 1$

$$\int \frac{a_0}{ax+b}\,dx = \frac{1}{a}\int \frac{a_0}{y}\,dy + k$$

$$y = ax+b$$

$$x = \frac{y-b}{a}$$

$$\boxed{dx = \frac{1}{a}dy}$$

$$= \left\{ \frac{a_0}{a}\log|y| + k \;\middle|\; k \in \mathbb{R} \right\} =$$

$$\left\{ \frac{a_0}{a}\log|ax+b| + k \right\}$$

$n = 2$

$$\int \frac{a_0 + a_1 x}{\boxed{ax^2+bx+c}}\,dx =$$

$$\Delta = b^2 - 4bc$$

$\Delta > 0$

$$= \frac{1}{a}\int \frac{a_0 + a_1 x}{(x-x_1)(x-x_2)}\,dx$$

where x_1, x_2 are the roots of ax^2+bx+c.

$$\boxed{\frac{a_0+a_1x}{(x-x_1)(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}}$$

$$a_0+a_1x = A(x-x_2)+B(x-x_1)$$

$$(*)\quad a_0+a_1x = (-x_2A-x_1B)+x(A+B).$$

P Q polinomials

$P(x)=Q(x) \quad x\in\mathbb{R}$

$\Updownarrow$

P and Q have the same coefficients.

So, by $(*)$

$$\begin{cases} -x_1A-x_2B = a_0 \\ A+B = a_1 \end{cases}$$

Solve it and determine A B

So we have to find

$$\int \frac{A}{x-x_1} + \frac{B}{x-x_2} =$$

$$= A \log|x - x_1| + B \log|x - x_2|.$$

$$\int \frac{5x+3}{x^2-3x+1}\,dx$$

by exercise.