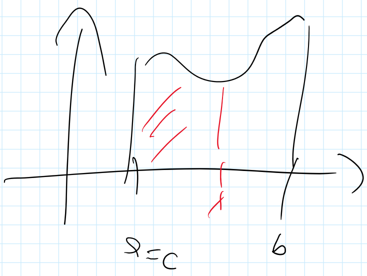
Recall of the Fundamental Theorem:

Let $f : [a,b] \to \mathbb{R}$ be continuous. Fix $c \in [a,b]$ and set

$$F_c(x) = \int_c^x f(t)\,dt. \quad \text{Then} \quad F_c'(x) = f(x) \quad \forall x \in [a,b].$$



Corollary Let $f : [a,b] \to \mathbb{R}$ be continuous and let F be any primitive of f (recall: $F'(x) = f(x)$ $\forall x \in [a,b]$). Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(=: F(x)\Big|_a^b \text{ or } =: [F(x)]_{x=a}^{x=b} \right)$$

NOTATION

Proof. Take $F_a(x) = \int_a^x f(t)\,dt$. We know that F_a is a primitive of f.

Therefore $F_a - F$ is constant on $[a,b]$, i.e. there exists $k \in \mathbb{R}$ such that $F_a(x) - F(x) = k$ $\forall x \in [a,b]$.

Observe that $F_a(a) = 0$, $F_a(b) = \int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = F_a(b) - F_a(a) = F(b) + \cancel{k} - (F(a) + \cancel{k})$$

$$= F(b) - F(a). \ \#$$

Corollary 2 Let $f : [a,b] \to \mathbb{R}$ be of class $\mathcal{C}^1$ (i.e. f is differentiable and f' is continuous). Then $\int_a^b f'(x)\,dx = f(b) - f(a)$.

and f' is continuous). Then $\int_a^b f'(x)dx = f(b)-f(a)$.

Proof: obvious because f is a primitive of f'. #

Definition (indefinite integral). We denote by

$$\int f(x)\,dx$$

the set of all primitives of f.

ex.: $\int \sin x \, dx = -\cos x + c, \; c \in \mathbb{R}$

Examples of "immediate" integrals:

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + c, \; c \in \mathbb{R}$$

$$(\alpha \neq -1) \qquad \int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \; c \in \mathbb{R}$$

$$\int \frac{1}{x}\,dx = \log|x| + c, \; c \in \mathbb{R}$$

Verification: $\frac{d}{dx}\log|x| \underset{x \neq 0}{=} \frac{1}{|x|}\operatorname{sgn}(x) = \frac{1}{x}$

where $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$

Thus, for $x \neq 0$, $\frac{\operatorname{sgn}(x)}{|x|} = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ \frac{-1}{-x} & \text{if } x<0 \end{cases} = \frac{1}{x}$

$$\int \left(\frac{2}{x} - x^{30} \right) dx = 2 \int \frac{1}{x}\, dx - \int x^{30} dx = 2 \log|x| - \frac{x^{31}}{31} + c, \; c \in \mathbb{R}$$

$$\int \cosh 2x \, dx = \int \frac{e^{2x} + e^{-2x}}{2} dx = \frac{e^{2x} - e^{-2x}}{4} + c = \frac{\sinh 2x}{2} + c, \; c \in \mathbb{R}$$

$$\int \sin x \cos x \, dx = \int \frac{1}{2} \sin 2x \, dx = -\frac{1}{4} \cos 2x + c, \; c \in \mathbb{R}$$

alternatively

$$\boxed{\frac{d}{dx} f^2(x) = 2 f(x) f'(x)} \longrightarrow = \int \frac{d}{dx} \left(\frac{1}{2} \sin^2 x \right) dx = \frac{1}{2} \sin^2 x + c, \; c \in \mathbb{R}$$

Observe that the set of functions $\left\{ -\frac{1}{4} \cos 2x + k, \; k \in \mathbb{R} \right\}$ and the set of functions $\left\{ \frac{1}{2} \sin^2 x + c, \; c \in \mathbb{R} \right\}$ coincide, as they must. Indeed

$$\cos 2x = 1 - 2\sin^2 x \quad \forall x, \text{ so that}$$

$$-\frac{1}{4} \cos 2x = -\frac{1}{4} (1 - 2\sin^2 x) = \frac{\sin^2 x}{2} - \frac{1}{4}.$$

Thus, the two sets do coincide.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d}{dx} \log(-\cos x)\, dx = -\log|\cos x| + c, \; c \in \mathbb{R}$$

$$\frac{d}{dx} \log|f(x)| \underset{f(x) \neq 0}{=} \frac{f'(x)\, \operatorname{sgn}(f(x))}{|f(x)|} = \frac{f'(x)}{f(x)}$$

$$\int \frac{x}{4x^2+1} dx = \frac{1}{8} \int \frac{8x}{4x^2+1} dx = \frac{1}{8} \log(4x^2+1) + c, \; c \in \mathbb{R}$$

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{2}\arctan\frac{x}{2} + c$$

$$\frac{d}{dx}\arctan\frac{x}{2} = \frac{1}{2}\,\frac{1}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{2}\,\frac{4}{x^2+4}$$

$$= \frac{2}{x^2+4}$$

The formula of integration by parts

Let $f, g : [a,b] \to \mathbb{R}$ be of class $\mathcal{C}^1$. Then

$$\underbrace{\int_a^b f(x)\,g'(x)\,dx} = \underbrace{f(x)\,g(x)\Big|_a^b} - \underbrace{\int_a^b f'(x)\,g(x)\,dx}.$$

Proof Observe that

$$\frac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)\,g(x) + f(x)\,g'(x)$$

Integrating

$$\int_a^b \frac{d}{dx}\big(f(x)g(x)\big)\,dx = \underbrace{\int_a^b f'(x)\,g(x)\,dx} + \underbrace{\int_a^b f(x)\,g'(x)\,dx}. \quad \#$$

$$\| $$

$$\underbrace{f(b)g(b) - f(a)g(a)}$$

examples

$$\int_0^\pi \underset{\substack{\| \\ f(x)}}{x}\ \underset{\substack{\| \\ g'(x)}}{\sin x}\,dx = x(-\cos x)\Big|_0^\pi - \int_0^\pi 1\,(-\cos x)\,dx$$

$$= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx$$

$$= \pi + \sin x\Big|_0^\pi = \pi$$

Warning: taking $x = g'(x)$ and $\sin x = f(x)$ is a bad choice, Actually

$$\int_0^\pi x \sin x \, dx = \frac{x^2}{2} \sin x \Big|_0^\pi - \underbrace{\int_0^\pi \frac{x^2}{2} \cos x \, dx}_{??}$$

not a good choice

"Indefinite version" of the formula of integration by parts:

$$\int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx$$

- $$\int \underset{f(x)}{x} \, \underset{g'(x)}{e^{2x}} \, dx = x \frac{e^{2x}}{2} - \int 1 \frac{e^{2x}}{2} \, dx = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

- $$\int \log x \, dx = \int \log x \cdot 1 \, dx = x \log x - \int \frac{1}{x} x \, dx$$

$$= x \log x - x + c$$

(check: $\frac{d}{dx}\left(x \log x - x\right) = \log x + \frac{x}{x} - 1 = \log x$)

- $$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx$$

$$= x \arctan x - \frac{1}{2} \log (1+x^2) + c$$

- $$\int x^2 e^{3x} \, dx = x^2 \frac{e^{3x}}{3} - \frac{2}{3} \int x \, e^{3x} \, dx$$

$$= x^2 \frac{e^{3x}}{3} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \frac{1}{3}\int e^{3x}\,dx\right]$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + c$$

- $$\int \sin^2 x\,dx = \int \sin x \,\sin x\,dx = -\sin x\cos x + \int \cos^2 x\,dx$$

$$= -\sin x\cos x + \int (1-\sin^2 x)\,dx$$

$$= -\sin x\cos x + x - \int \sin^2 x\,dx$$

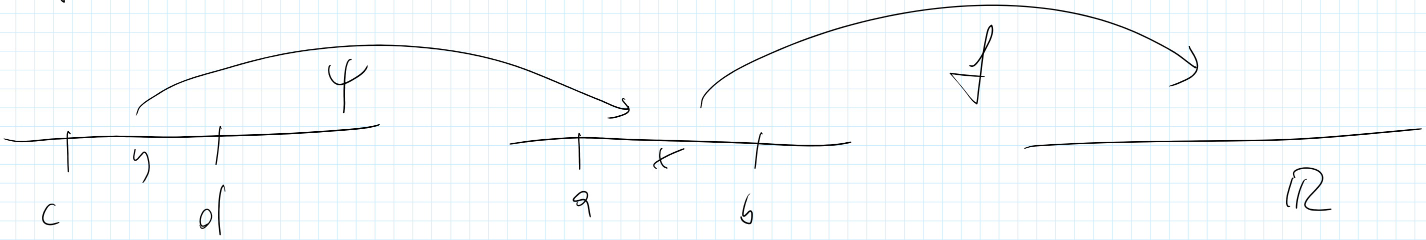
Thus: $$2\int \sin^2 x\,dx = -\sin x\cos x + x + c$$

$$\int \sin^2 x\,dx = \frac{1}{2}\left(x - \sin x\cos x\right) + c$$

For you: compute $\int x \arctan x\,dx$, $\int \arcsin x\,dx$

Integration by substitution (change of variable formula)

Proposition Let $f:[a,b]\to\mathbb{R}$ be continuous and let $\varphi:[c,d]\to[a,b]$ be a bijection that is of class C^1 together with its inverse.



Then
$$\int_a^b f(x)\,dx = \int_c^d f(\varphi(y))\,|\varphi'(y)|\,dy = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(y))\,\varphi'(y)\,dy$$

[No proof: it is a consequence of the chain rule]

How to use this formula:
$$\int_a^b f(x)\,dx \underset{\substack{x=\varphi(y)}}{=} \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(y))\,\varphi'(y)\,dy$$

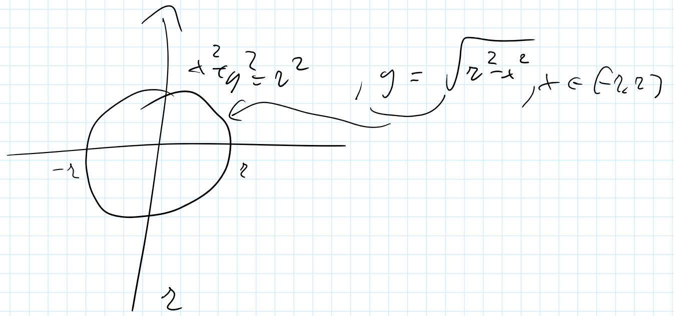
$x = \varphi(y)$

$\frac{dx}{dy} = \varphi'(y) \xrightarrow{\text{FORMALLY}} dx = \varphi'(y)\,dy$

$a = \varphi(\varphi^{-1}(a))$

Examples:

1) Area of a disk of radius $r > 0$



$$\text{Area}\,\{(x,y) : x^2+y^2 \le r^2\} =$$

$$= 2\int_{-r}^{r} \sqrt{r^2-x^2}\,dx = 2r\int_{-r}^{r} \sqrt{1-\left(\frac{x}{r}\right)^2}\,dx$$

$$\underset{\uparrow}{=} 2r\int_{-1}^{1} \sqrt{1-y^2}\; r\,dy = 2r^2\int_{-1}^{1} \sqrt{1-y^2}\,dy$$

$\frac{x}{r} = y$

$x = ry$

$dx = r\,dy$

$$\underset{\uparrow}{=} 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\vartheta}\,\cos\vartheta\,d\vartheta$$

$y = \sin\vartheta$, $\vartheta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$dy = \cos\vartheta\,d\vartheta$

$$= 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2\vartheta}\, \cos\vartheta \, d\vartheta = 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\vartheta \, d\vartheta$$

$$= 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\vartheta)\, d\vartheta = 2r^2 \left(\vartheta - \frac{1}{2}\left(\vartheta - \sin\vartheta\cos\vartheta\right)\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= r^2 \left(\vartheta + \sin\vartheta\cos\vartheta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi r^2$$

For you: compute the area of the ellipse $\left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$

$(a, b > 0)$; Result $= \pi a b$

<u>Further examples</u>: Let us compute $\int x \sqrt{4x-1} \, dx$ in two different ways.

(of course $4x-1 \geq 0$)

a) by change of variables

$$\int x \sqrt{4x-1} \, dx \underset{\uparrow}{=} \int \frac{t^2+1}{4} \cdot t \cdot \frac{t}{2} \, dt = \frac{1}{8} \int (t^4 + t^2)\, dt$$

$4x-1 = t^2$

$x = \frac{t^2+1}{4}$

$dx = \frac{t}{2} \, dt$

$$= \frac{1}{8} \left(\frac{t^5}{5} + \frac{t^3}{3} \right) + C$$

$$= \frac{1}{8} \left(\frac{(4x-1)^{\frac{5}{2}}}{5} + \frac{(4x-1)^{\frac{3}{2}}}{3} \right) + C$$

b) by parts

$$\int x\sqrt{4x-1}\,dx \underset{\uparrow}{=} \frac{x(4x-1)^{\frac{3}{2}}}{6} - \frac{1}{6}\int (4x-1)^{\frac{3}{2}}\,dx$$

$$\sqrt{4x-1} = \frac{d}{dx}\frac{1}{6}(4x-1)^{\frac{3}{2}}$$

$$= \frac{x(4x-1)^{\frac{3}{2}}}{6} - \frac{1}{60}(4x-1)^{\frac{5}{2}} + c$$

Verify that the two primitives are equal up to a constant

$$- \quad \int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx \underset{\uparrow}{=} \int \frac{t^2}{\sqrt{1+t}}\,\frac{dt}{t} = \int \frac{t}{\sqrt{1+t}}\,dt$$

$e^x = t$

$x = \log t$

$dx = \frac{dt}{t}$

$$= \int \frac{t+1}{\sqrt{t+1}}\,dt - \int \frac{dt}{\sqrt{t+1}}$$

$$= \int \sqrt{t+1}\,dt - \int \frac{dt}{\sqrt{1+t}} = \frac{2}{3}(t+1)^{\frac{3}{2}} - 2\sqrt{1+t} + c$$

$$= \frac{2}{3}(e^x+1)^{\frac{3}{2}} - 2\sqrt{1+e^x} + c$$