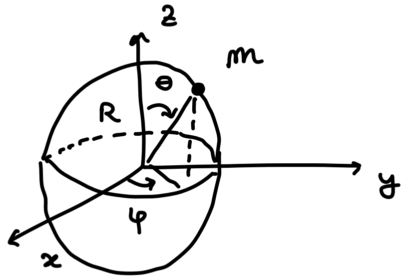
Lesson 30 – 06/12/2022

Spherical pendulum.

Point P, m contrained on a sphere and subjected to the gravity



$\varphi \in [0, 2\pi)$
$\vartheta \in (0, \pi)$

$$\begin{cases} x = R\sin\vartheta\cos\varphi \\ y = R\sin\vartheta\sin\varphi \\ z = R\cos\vartheta \end{cases} \Rightarrow \begin{cases} \dot{x} = R\dot{\vartheta}\cos\vartheta\cos\varphi - R\dot{\varphi}\sin\vartheta\sin\varphi \\ \dot{y} = R\dot{\vartheta}\cos\vartheta\sin\varphi + R\dot{\varphi}\sin\vartheta\cos\varphi \\ \dot{z} = -R\dot{\vartheta}\sin\vartheta \end{cases}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dots = R^2\dot{\vartheta}^2\cos^2\vartheta\cos^2\varphi + R^2\dot{\varphi}^2\sin^2\vartheta\sin^2\varphi$$
$$+ R^2\dot{\vartheta}^2\cos^2\vartheta\sin^2\varphi + R^2\dot{\varphi}^2\sin^2\vartheta\cos^2\varphi + R^2\dot{\vartheta}^2\sin^2\vartheta =$$
$$= R^2\dot{\vartheta}^2\cos^2\vartheta + R^2\dot{\varphi}^2\sin^2\vartheta + R^2\dot{\vartheta}^2\sin^2\vartheta =$$
$$= R^2\dot{\vartheta}^2 + R^2\dot{\varphi}^2\sin^2\vartheta$$

$$L = L(\vartheta, \varphi, \dot{\vartheta}, \dot{\varphi}) = \frac{mR^2}{2}(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2) - mgR\cos\vartheta$$

we can divide by mR^2

$$\frac{1}{2}(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2) - \underbrace{\frac{g}{R}}_{= K > 0}\cos\vartheta$$

$$L = \frac{1}{2}(\dot{\vartheta}^2 + \sin^2\vartheta\,\dot{\varphi}^2) - K\cos\vartheta$$
$$= L(\vartheta, \dot{\vartheta}, \dot{\varphi})$$

But φ is a cyclic coord.

$$\frac{\partial L}{\partial\dot{\varphi}} = J = \sin^2\vartheta\,\dot{\varphi} \iff \dot{\varphi} = J/\sin^2\vartheta$$

$(J \neq 0)$

Reduced lagrangian:

$$L_R(\vartheta,\dot\vartheta) = \frac{1}{2}\Big(\dot\vartheta^2 + \underbrace{\cancel{\sin^2\vartheta}\frac{J^2}{\sin^{\cancel{4}2}\vartheta}}\Big) - k\cos\vartheta - J\underbrace{\Big(\frac{J}{\sin^2\vartheta}\Big)}$$

$$= \frac{1}{2}\dot\vartheta^2 - \underbrace{\Big(k\cos\vartheta + \frac{J^2}{2\sin^2\vartheta}\Big)}_{V_R(\vartheta)}$$

$\hookrightarrow$ lagrangian of a 1-dim conservative dynamical system.
where $\vartheta \in (0,\pi)$. We can draw the phase-portrait!

$$\ddot\vartheta = -V_R'(\vartheta) \qquad (J \neq 0)$$

Graph of $V_R(\vartheta) = k\cos\vartheta + \dfrac{J^2}{2\sin^2\vartheta}$

$$\lim_{\vartheta\to 0^+} V_R(\vartheta) = \lim_{\vartheta\to\pi^-} V_R(\vartheta) = +\infty.$$

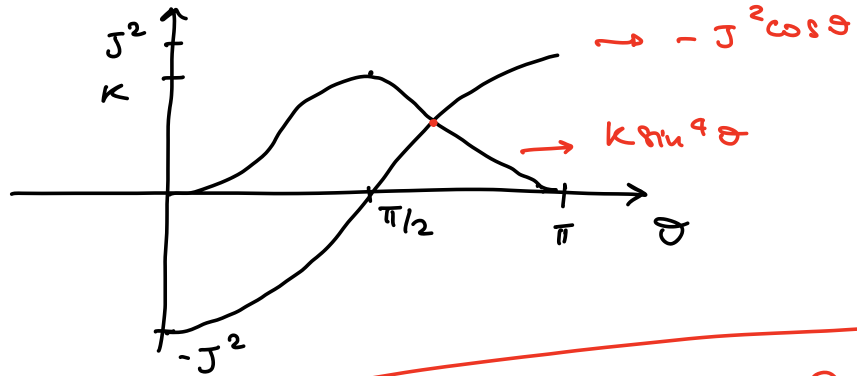
$\Rightarrow$ $V_R(\vartheta)$ has AT LEAST 1 critical point $\in (0,\pi)$, which is a minimum. Is it the unique critical point?
We ~~study~~ the $V_R'(\vartheta)$.

$$V_R'(\vartheta) = -k\sin\vartheta - \frac{\cancel{4}J^2\cos\vartheta\,\cancel{\sin\vartheta}}{\cancel{4}\sin^{\cancel{4}3}\vartheta} =$$

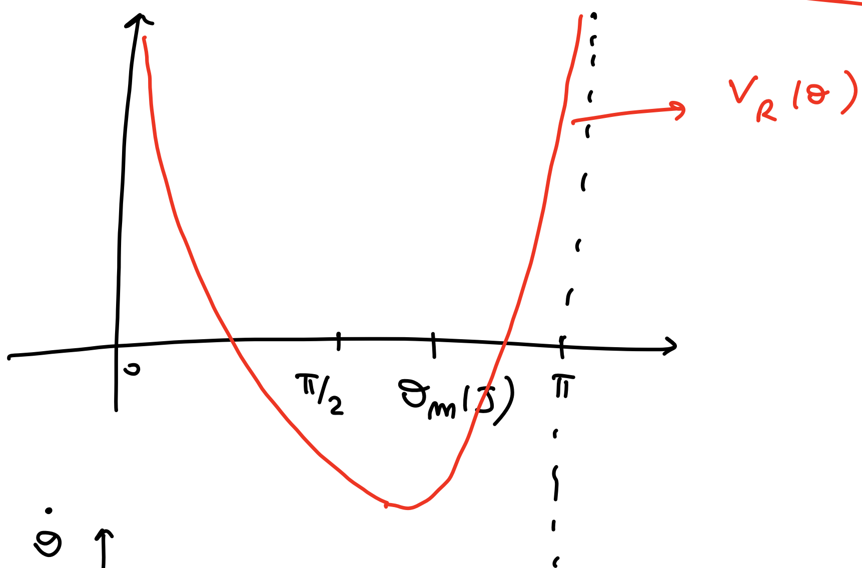
$$= -k\sin\vartheta - \frac{J^2\cos\vartheta}{\sin^3\vartheta} =$$

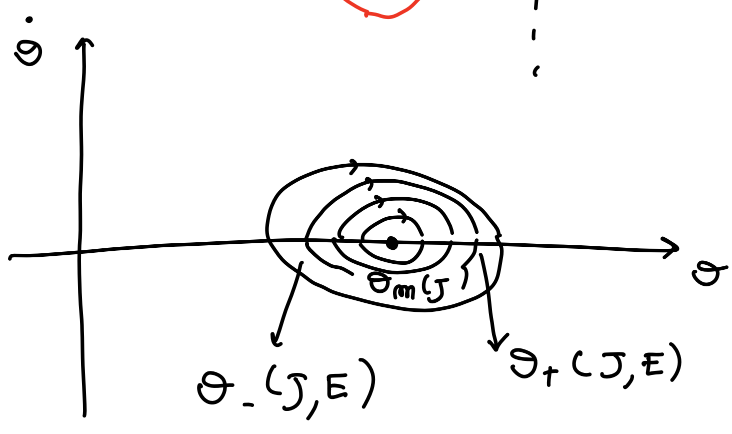
$$= \underbrace{-\frac{1}{\sin^3\vartheta}}_{\vartheta\in(0,\pi)}\left[k\sin^4\vartheta + J^2\cos\vartheta\right]$$

$$V_R'(\vartheta) = 0 \iff k\sin^4\vartheta = -J^2\cos\vartheta$$



$\Rightarrow$ $V_R(\theta)$ has an unique critical point $\theta_m(J) \in \left]\pi/2, \pi\right[$





For $J \neq 0$, the reduced system has an unique stable equilibrium and all other motions are periodic, with period $T = T(E,J)$.

<u>Re-construction</u> of the dynamics for the original system.

Recall that $\dot\varphi = J / \sin^2\vartheta$.

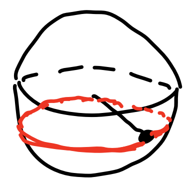
Therefore, if $t \mapsto \vartheta(t)$ is a (periodic) solution of the reduced system, then

$$\varphi(t) = \varphi(0) + \int_0^t \frac{J}{\sin^2[\vartheta(s)]}\, ds$$

Equilibrium:

$\vartheta(t) \equiv \vartheta_m(J)$, $\varphi(t) = \varphi_0 + \dfrac{Jt}{\sin^2[\vartheta_m(J)]}$

$\Rightarrow$ The pendulum rotates with constant angular velocity $J / \sin^2\vartheta_m(J)$.

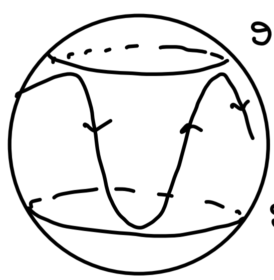


Some qualitative properties of other motions.

1) The coord. ϑ follows a periodic motion between a minimal value ϑ_- and a maximal value ϑ_+

2) Moreover, $\dot\varphi(t) = J / \underbrace{\sin^2(\vartheta(t))}_{>0}$ $\nearrow >0$ if $J>0$, $\searrow <0$ if $J<0$

$\Rightarrow$ $\varphi(t)$ is monotonically increasing (or decreasing).



3) Natural question is: The motions of the spherical pendulum are periodic? ($\Leftrightarrow$ trajectories are closed??)

it can be proved that, varying E and J, the motion continues to change from periodic (closed trajectories) to aperiodic (dense trajectories).

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For previous problem, coordinates ϑ, φ are ok. However, for examples, in order to study small oscillations of the spherical pendulum around the stable eq. (south pole) we need to change coordinates.

$$\begin{cases} x \\ y \\ z = -\sqrt{R^2 - x^2 - y^2} \end{cases}$$

↳ Linearized eqs. of the spherical pendulum around the south pole.

Linearization around an equilibrium for a mechanical Lagrangian is given by Lagrange eqs for

$$L^0(x, y, \dot{x}, \dot{y}) = \frac{1}{2} a(0,0) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \frac{1}{2} \operatorname{Hess} V(0,0) \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{z} = \frac{x\dot{x} + y\dot{y}}{\sqrt{R^2 - x^2 - y^2}}$$

$$\Rightarrow K = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \frac{(x\dot{x} + y\dot{y})^2}{(R^2 - x^2 - y^2)} \right) =$$

$$= \frac{1}{2} m \left[\dot{x}^2 \left(1 + \frac{x^2}{R^2 - x^2 - y^2} \right) + \dot{y}^2 \left(1 + \frac{y^2}{R^2 - x^2 - y^2} \right) + \frac{2xy\dot{x}\dot{y}}{(R^2 - x^2 - y^2)} \right]$$

$$V(x,y) = mgz = -mg\sqrt{R^2-x^2-y^2}$$

$$L = K - V$$

$$a(0,0) = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nabla V(x,y) = mg\left(\frac{x}{(R^2-x^2-y^2)^{1/2}}, \frac{y}{(R^2-x^2-y^2)^{1/2}}\right)$$

$$\mathrm{Hess}\, V(0,0) = \frac{mg}{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{L^\circ(x,y,\dot{x},\dot{y}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{mg}{2R}(x^2+y^2)}$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L^\circ}{\partial \dot{x}}\right) - \dfrac{\partial L^\circ}{\partial x} = 0 \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial L^\circ}{\partial \dot{y}}\right) - \dfrac{\partial L^\circ}{\partial y} = 0 \end{cases} \qquad \underbrace{\begin{cases} m\ddot{x} = -\dfrac{mg}{R}x \\[2ex] m\ddot{y} = -\dfrac{mg}{R}y \end{cases}}$$

are the linearized eqs. of the spherical pendulum around $(0,0) \to$ south pole (stable eq.).

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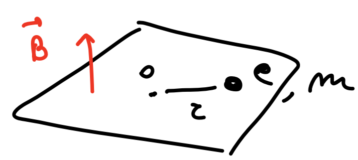
Potentials depending on velocities: <u>the magnetic stabilization</u>

- Electric charge on the plane, subjected to a conservative repulsive force :



$$\vec{F} = K r \vec{e}_r \quad , \quad r^2 = x^2 + y^2$$

$$K > 0$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} K r^2$$

$$V(r) = -\frac{1}{2} K r^2$$

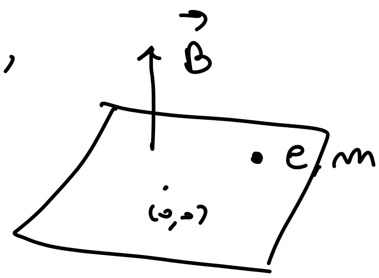
$(\underbrace{0,0}_{x,y}, \underbrace{0,0}_{\dot{x},\dot{y}})$ is an unstable equilibrium $\forall K \geq 0$.

$\underline{K=0}$: unstability of the free particle.

$\underline{K>0}$: unstability of the harmonic repeller.

- We add a magnetic field,

$$\vec{B} = B \vec{e}_z$$

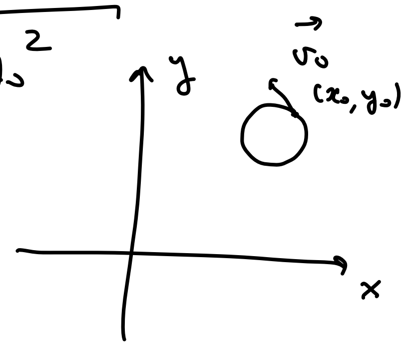


$$\Rightarrow \vec{F} = -e \vec{B} \wedge \vec{v} = e B \vec{v} \wedge \vec{e}_z$$

- $\boxed{K=0}$ → A charge on a magnetic field.

In such a case, the motion through (x_0, y_0) is circular with radius:

$$R = \frac{m v_0}{|eB|} \quad , \quad v_0 = \sqrt{\dot{x}_0^2 + \dot{y}_0^2}$$



$\Rightarrow$ Every point $(\overbrace{x_0, y_0}, \overbrace{0, 0}^{\text{Velocity.}})$ is stable.

• $\boxed{k>0}$ The origin is stable if

$$\boxed{B^2 > 4\frac{km}{e^2}}$$

New Lagrangian:

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}kz^2 - \frac{1}{2}e\vec{B}\wedge\vec{v}\cdot\vec{r}$$

Now

$$\frac{1}{2}e\vec{B}\wedge\vec{v}\cdot\vec{r} = \frac{1}{2}eB\vec{e}_z\wedge\vec{v}\cdot\vec{r} =$$

$$= \underbrace{\frac{1}{2}\frac{e\,m}{m}B\vec{e}_z}_{m\vec{\omega}}\wedge\vec{v}\cdot\vec{r} =$$

$$= \underbrace{m\vec{\omega}\wedge\vec{v}\cdot\vec{r}}_{\text{Generalized potential for a Coriolis force.}}$$

$$\boxed{\vec{\omega} = \frac{eB\vec{e}_z}{2m}}$$

"angular velocity"

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \underbrace{+\frac{1}{2}kz^2 - \frac{1}{2}m\omega^2 z^2}_{= -V_0(z)} \underbrace{+\frac{1}{2}m\omega^2 z^2}_{-\text{Centrifugal potential}} \underbrace{- m\vec{\omega}\wedge\vec{v}\cdot\vec{r}}_{-\text{Coriolis generalized potential}}$$

In the inertial system:

$$\tilde{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}kr^2 - \frac{1}{2}m\omega^2 z^2$$

$$V_0(r) = \frac{1}{2}(m\omega^2 - k)\, r^2$$

$\exists!$ equilibrium $r = 0$ $\quad ((x,y) = (0,0))$

which results stable when

$$m\omega^2 - k > 0$$

$$\Leftrightarrow \cancel{m}\, \frac{e^2 B^2}{4 m^{\cancel{2}}} - k > 0$$

$$\Leftrightarrow B^2 > \frac{k \cdot 4m}{e^2}$$

<u>Remark</u> This stability is weak, indeed it is destroyed by adding a friction. Because - adding a friction - $\vec{v}$ decreases $\Rightarrow$ $\vec{F} = e\vec{B}\, \vec{v} \wedge \vec{e}_z$ decreases itself.

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